THE CHOWLA-SELBERG FORMULA FOR ABELIAN CM FIELDS AND FALTINGS HEIGHTS

ADRIAN BARQUERO-SANCHEZ AND RIAD MASRI

Abstract. In this paper we establish a Chowla-Selberg formula for abelian CM fields. This is an identity which relates values of a Hilbert modular function at CM points to values of Euler's gamma function Γ and an analogous function Γ_2 at rational numbers. We combine this identity with work of Colmez to relate the CM values of the Hilbert modular function to Faltings heights of CM abelian varieties. We also give explicit formulas for products of exponentials of Faltings heights, allowing us to study some of their arithmetic properties using the Lang-Rohrlich conjecture.

1. Introduction and statement of results

1.1. **Overview.** The Chowla-Selberg formula [CS1, CS2] is a remarkable identity which relates values of the Dedekind eta function at CM points to values of Euler's gamma function Γ at rational numbers. This formula arises in connection with many topics in number theory, including elliptic curves, L–functions, modular forms, and transcendence. For a very nice discussion, see Zagier [Z, Section 6.3]. In this paper we will establish a Chowla-Selberg formula for abelian CM fields. This is an identity which relates values of a Hilbert modular function at CM points to values of Γ and an analogous function Γ_2 at rational numbers. The function Γ_2 was studied extensively by Deninger [D] in his work on the Chowla-Selberg formula for real quadratic fields. We will combine our Chowla-Selberg formula for abelian CM fields with a theorem of Colmez [Col] which relates Faltings heights of CM abelian varieties to logarithmic derivatives of Artin L–functions to give a geometric interpretation of the CM values. Using this circle of ideas, we will also give explicit formulas for products of exponentials of Faltings heights, allowing us to study some of their arithmetic properties using the Lang-Rohrlich conjecture, which concerns algebraic relations among values of Γ at rational numbers. We note that there has recently been a great amount of interest in formulas for CM values of Hilbert modular functions. Some examples occur in the work of Bruinier-Yang [BY, BY2, BY3] and Bruinier-Kudla-Yang [BKY], which is related to Borcherds products and the seminal work of Gross-Zagier [GZ] on factorization of differences of singular moduli. See also the work of Yang [Ya, Ya2, Ya3], which reveals new connections between the Chowla-Selberg formula, Faltings heights, and arithmetic intersection theory.

1.2. **The Chowla-Selberg formula.** We begin by reviewing the classical Chowla-Selberg formula (see e.g. [W, Chapter IX]). Let $\Delta = f^2 d$ be a fundamental discriminant where $f > 0$ and d is square-free. Let $K = \mathbb{Q}(\sqrt{d})$ be a quadratic field of discriminant Δ, $\mathcal{O}_K$ be the ring of integers, $\mathrm{CL}(K)$ be the ideal class group, h_d be the class number, $w_d = \#\mathcal{O}_K^\times$ be the number of units (for $d < 0$), ε_d be the fundamental unit (for $d > 0$), and $\chi_d(\cdot) = \left(\frac{\Delta}{\cdot}\right)$ be the Kronecker symbol associated to K. Assume now that $d < 0$. Given an ideal class $C \in \mathrm{CL}(K)$, one may choose a primitive integral ideal $\mathfrak{a} \in C^{-1}$ such that

$$\mathfrak{a} = \mathbb{Z}a + \mathbb{Z}\left(\frac{-b+\sqrt{\Delta}}{2}\right), \quad a, b \in \mathbb{Z}$$

2010 *Mathematics Subject Classification.* 11F41, 11R42, 11G50.

Key words and phrases. Chowla-Selberg formula, CM point, Faltings height, Hilbert modular function.

where $a = N_{K/\mathbb{Q}}(\mathfrak{a})$ is the norm of $\mathfrak{a}$ and b satisfies $b^2 \equiv \Delta \mod 4a$. Then

$$\tau_{\mathfrak{a}} = \frac{-b+\sqrt{\Delta}}{2a}$$

is a CM point in the complex upper half-plane $\mathbb{H}$ which corresponds to the inverse class $[\mathfrak{a}] = C^{-1}$.

The Chowla-Selberg formula is obtained by comparing two different expressions for the Dedekind zeta function $\zeta_K(s)$. One has the classical identity

$$\zeta_K(s) = \frac{2}{w_d}\zeta(2s)\left(\frac{2}{\sqrt{|\Delta|}}\right)^s \sum_{[\mathfrak{a}]\in \mathrm{CL}(K)} E(\tau_{\mathfrak{a}}, s),$$

where

$$E(z,s) := \sum_{M\in\Gamma_\infty\backslash \mathrm{SL}_2(\mathbb{Z})} \mathrm{Im}(Mz)^s, \quad z\in\mathbb{H}, \quad \mathrm{Re}(s)>1$$

is the non-holomorphic Eisenstein series for $\mathrm{SL}_2(\mathbb{Z})$. On the other hand, one has the well-known factorization

$$\zeta_K(s) = \zeta(s)L(\chi_d, s),$$

where $L(\chi_d,s)$ is the Dirichlet L-function associated to χ_d. Comparing these expressions and making the shift $s\mapsto (s+1)/2$ yields

$$\sum_{[\mathfrak{a}]\in \mathrm{CL}(K)} E\left(\tau_{\mathfrak{a}}, \frac{s+1}{2}\right) = \frac{w_d}{2}\left(\frac{\sqrt{|\Delta|}}{2}\right)^{\frac{s+1}{2}} \frac{\zeta(\frac{s+1}{2})}{\zeta(s+1)} L\left(\chi_d, \frac{s+1}{2}\right). \tag{1.1}$$

Now, one has the "renormalized" Kronecker limit formula

$$E\left(z, \frac{s+1}{2}\right) = 1 + \log(G(z))(s+1) + O((s+1)^2), \tag{1.2}$$

where

$$G(z) := \sqrt{\mathrm{Im}(z)}|\eta(z)|^2$$

and

$$\eta(z) := q^{1/24}\prod_{n=1}^{\infty}(1-q^n), \quad q := e^{2\pi i z}, \quad z\in\mathbb{H}$$

is the Dedekind eta function, a weight 1/2 cusp form for $\mathrm{SL}_2(\mathbb{Z})$. Substitute (1.2) into the left hand side of (1.1), calculate the Taylor expansion of the right hand side of (1.1) at $s=-1$, differentiate both sides of the resulting identity with respect to s, and evaluate at $s=-1$ to get

$$\sum_{[\mathfrak{a}]\in \mathrm{CL}(K)} \log(G(\tau_{\mathfrak{a}})) = \frac{w_d}{2}L(\chi_d,0)\left\{\log\left(\frac{\sqrt{|\Delta|}}{2}\right) - \frac{\zeta'(0)}{\zeta(0)} + \frac{L'(\chi_d,0)}{L(\chi_d,0)}\right\}. \tag{1.3}$$

Recall the evaluation

$$-\frac{\zeta'(0)}{\zeta(0)} = -\log(2\pi), \tag{1.4}$$

and the class number formula

$$L(\chi_d,0) = \frac{2h_d}{w_d}. \tag{1.5}$$

To evaluate $L'(\chi_d, 0)$, one uses the decomposition

$$L(\chi_d, s) = |\Delta|^{-s} \sum_{k=1}^{|\Delta|} \chi_d(k) \zeta\left(s, \frac{k}{|\Delta|}\right), \tag{1.6}$$

where

$$\zeta(s, w) := \sum_{n=0}^{\infty} \frac{1}{(n+w)^s}, \quad \mathrm{Re}(w) > 0, \quad \mathrm{Re}(s) > 1$$

is the Hurwitz zeta function. Lerch [Le] showed that

$$\zeta(s, x) = \frac{1}{2} - x + \log\left(\frac{\Gamma(x)}{\sqrt{2\pi}}\right) s + O(s^2), \quad x > 0 \tag{1.7}$$

where

$$\Gamma(s) := \int_0^{\infty} t^{s-1} e^{-t} dt$$

is Euler's gamma function. Substitute (1.7) into (1.6), then differentiate to get

$$L'(\chi_d, 0) = -\log(|\Delta|) L(\chi_d, 0) + \sum_{k=1}^{|\Delta|} \chi_d(k) \log\left\{\Gamma\left(\frac{k}{|\Delta|}\right)\right\}. \tag{1.8}$$

Finally, substitute (1.4), (1.5) and (1.8) into (1.3), then exponentiate to obtain the Chowla-Selberg formula

$$\prod_{[\mathfrak{a}] \in \mathrm{CL}(K)} G(\tau_{\mathfrak{a}}) = \left(\frac{1}{4\pi\sqrt{|\Delta|}}\right)^{\frac{h_d}{2}} \prod_{k=1}^{|\Delta|} \Gamma\left(\frac{k}{|\Delta|}\right)^{\frac{w_d \chi_d(k)}{4}}. \tag{1.9}$$

1.3. **Statement of the main results.** To establish a Chowla-Selberg formula for abelian CM fields, we will follow the basic structure of the argument just described.

The following facts concerning Hilbert modular varieties and CM points are explained in detail in Sections 3 and 4.

Let $F/\mathbb{Q}$ be a totally real field of degree n. Let $\mathcal{O}_F$ be the ring of integers, $\mathcal{O}_F^{\times}$ be the group of units, d_F be the absolute value of the discriminant, and $\zeta_F(s)$ be the Dedekind zeta function. Let $z = (z_1, \ldots, z_n) \in \mathbb{H}^n$. The Hilbert modular group $\mathrm{SL}_2(\mathcal{O}_F)$ acts componentwise on $\mathbb{H}^n$ by linear fractional transformations.

Let E be a CM extension of F and $\Phi = \{\sigma_1, \ldots, \sigma_n\}$ be a CM type for E. Let h_E be the class number of E, and assume that F has narrow class number 1. Given an ideal class $C \in \mathrm{CL}(E)$, let $z_{\mathfrak{a}}$ be a CM point corresponding to the inverse class $[\mathfrak{a}] = C^{-1}$. To ease notation, we identify the CM point $z_{\mathfrak{a}}$ with its image $\Phi(z_{\mathfrak{a}}) \in \mathbb{H}^n$ under the CM type Φ. Let

$$\mathcal{CM}(E, \Phi, \mathcal{O}_F) := \{z_{\mathfrak{a}} : \ [\mathfrak{a}] \in \mathrm{CL}(E)\}$$

be a set of CM points of type (E, Φ). This is a CM zero-cycle on the Hilbert modular variety $\mathrm{SL}_2(\mathcal{O}_F)\backslash\mathbb{H}^n$.

We will establish the following analog of (1.3),

$$\sum_{[\mathfrak{a}] \in \mathrm{CL}(E)} \log\left(H(z_{\mathfrak{a}})\right) = \frac{h_E}{2}\left\{\log\left(\frac{\sqrt{d_E}}{2^n d_F}\right) - \frac{1}{n}\frac{\zeta_F^{(n)}(0)}{\zeta_F^{(n-1)}(0)} + \frac{L'(\chi_{E/F}, 0)}{L(\chi_{E/F}, 0)}\right\}, \tag{1.10}$$

where $H : \mathbb{H}^n \to \mathbb{R}^+$ is a $\mathrm{SL}_2(\mathcal{O}_F)$-invariant function analogous to $G(z)$ which arises from a renormalized Kronecker limit formula for the non-holomorphic Hilbert modular Eisenstein series (see Section 3, and in particular, equation (3.8)), and $L(\chi_{E/F}, s)$ is the L–function of the quadratic character $\chi_{E/F}$ associated by class field theory to the CM extension E/F.

Assume now that E is *abelian* over $\mathbb{Q}$. Then $F \subset E \subset \mathbb{Q}(\zeta_N)$ for some primitive N-th root of unity $\zeta_N := e^{2\pi i/N}$. Let H_E (resp. H_F) be the subgroup of $G_N := \mathrm{Gal}(\mathbb{Q}(\zeta_N)/\mathbb{Q})$ which fixes E (resp. F). Using the isomorphism $G_N \cong (\mathbb{Z}/N\mathbb{Z})^\times$, one defines the group of Dirichlet characters associated to E (resp. F) by

$$X_E := \{\chi \in \widehat{(\mathbb{Z}/N\mathbb{Z})^\times} : \ \chi_{|H_E} \equiv 1\}$$

(resp. X_F). Clearly, we have $H_E \leq H_F$ and $X_F \leq X_E$.

Given a Dirichlet character $\chi \in X_E$, let $L(\chi, s)$ denote the L–function of the primitive Dirichlet character of conductor c_χ which induces χ. The Gauss sum of $\chi \in X_E$ is defined by

$$\tau(\chi) := \sum_{k=1}^{c_\chi} \chi(k)\zeta_{c_\chi}^k, \quad \zeta_{c_\chi} := e^{2\pi i/c_\chi}.$$

We will establish the identity

$$\frac{L'(\chi_{E/F}, s)}{L(\chi_{E/F}, s)} = \sum_{\chi \in X_E \setminus X_F} \frac{L'(\chi, s)}{L(\chi, s)},$$

hence to evaluate the logarithmic derivative of $L(\chi_{E/F}, s)$ at $s = 0$, we must evaluate $L'(\chi, 0)$ for $\chi \in X_E \setminus X_F$. We can express $L'(\chi, 0)$ in terms of values of $\log(\Gamma(s))$ at rational numbers as in (1.8).

On the other hand, we will reduce the evaluation of the logarithmic derivative of $\zeta_F^{(n-1)}(s)$ at $s = 0$ to the evaluation of $L'(\chi, 1)$ for nontrivial $\chi \in X_F$. Because each $\chi \in X_F$ is even, $L'(\chi, 1)$ cannot be expressed in terms of values of $\log(\Gamma(s))$ at rational numbers (this is due to the sign of the functional equation for $L(\chi, s)$ when χ is even). However, Deninger [D] showed how to evaluate $L'(\chi, 1)$ in terms of values of the function

$$R(w) := \partial_s^2 \zeta(0, w), \quad \mathrm{Re}(w) > 0$$

at rational numbers. The function $R(w)$ is analogous to $\log(\Gamma(s)/\sqrt{2\pi})$, as we now explain.

Consider the Taylor expansion

$$\zeta(s, x) = \frac{1}{2} - x + \log\left(\frac{\Gamma(x)}{\sqrt{2\pi}}\right) s + R(x)s^2 + O(s^3), \quad x > 0.$$

By the Bohr-Mollerup theorem, $\log(\Gamma(x)/\sqrt{2\pi})$ is the unique function $f : \mathbb{R}^+ \to \mathbb{R}$ such that

$$f(x+1) - f(x) = \log(x),$$

$f(1) = \zeta'(0) = -\log(\sqrt{2\pi})$, and $f(x)$ is convex on $\mathbb{R}^+$. Using properties of the Hurwitz zeta function, one can show that $\partial_s \zeta(0, x)$ also satisfies these three conditions, hence by uniqueness, one recovers Lerch's identity

$$\partial_s \zeta(0, x) = \log\left(\frac{\Gamma(x)}{\sqrt{2\pi}}\right).$$

Note that using the limit

$$\Gamma(x) = \lim_{n\to\infty} \frac{n! n^x}{x(x+1)\cdots(x+n)}, \quad x > 0$$

one has

$$\log\left(\frac{\Gamma(x)}{\sqrt{2\pi}}\right) = \lim_{n\to\infty}\left(\zeta'(0) + x\log(n) - \log(x) - \sum_{k=1}^{n-1}(\log(x+k) - \log(k))\right). \tag{1.11}$$

Deninger [D, Theorem 2.2] proved a similar result for the functions $\partial_s^\alpha \zeta(0,x), \alpha \in \mathbb{Z}^+$, by modeling the proof of Lerch's identity just described. In particular, for $\alpha = 2$ he proved that $R(x)$ is the unique function $R : \mathbb{R}^+ \to \mathbb{R}$ such that

$$R(x+1) - R(x) = \log^2(x),$$

$R(1) = -\zeta''(0)$, and $R(x)$ is convex on (e, ∞). He also proved the following analog of (1.11),

$$R(x) = \lim_{n\to\infty} \left(-\zeta''(0) + x\log^2(n) - \log^2(x) - \sum_{k=1}^{n-1} \left(\log^2(x+k) - \log^2(k)\right) \right).$$

Define the function

$$\Gamma_2(w) := \exp(R(w)), \quad \mathrm{Re}(w) > 0$$

which is analogous to $\Gamma(s)/\sqrt{2\pi}$. Note that $\Gamma_2(w)$ does not extend to a meromorphic function on $\mathbb{C}$ (see e.g. [D, Remark (2.4)]).

We can now state our Chowla-Selberg formula for abelian CM fields.

Theorem 1.1. *Let $F/\mathbb{Q}$ be a totally real field of degree n with narrow class number 1. Let E/F be a CM extension with $E/\mathbb{Q}$ abelian. Let Φ be a CM type for E and*

$$\mathcal{CM}(E, \Phi, \mathcal{O}_F) = \{z_{\mathfrak{a}} : \ [\mathfrak{a}] \in \mathrm{CL}(E)\}$$

be a set of CM points of type (E, Φ). Then

$$\prod_{[\mathfrak{a}]\in \mathrm{CL}(E)} H(z_{\mathfrak{a}}) = c_1(E, F, n) \prod_{\chi\in X_E\setminus X_F} \prod_{k=1}^{c_\chi} \Gamma\left(\frac{k}{c_\chi}\right)^{\frac{h_E\chi(k)}{2L(\chi,0)}} \prod_{\substack{\chi\in X_F\\ \chi\neq 1}} \prod_{k=1}^{c_\chi} \Gamma_2\left(\frac{k}{c_\chi}\right)^{\frac{h_E\tau(\chi)\overline{\chi}(k)}{2c_\chi L(\chi,1)}},$$

where

$$c_1(E, F, n) := \left(\frac{d_F}{2^{n+1}\pi\sqrt{d_E}}\right)^{\frac{h_E}{2}}.$$

Remark 1.2. Given a triple (E, F, Φ) satisfying the hypotheses of Theorem 1.1, one can obtain explicit examples by determining the group of characters X_E (resp. X_F) and a set of CM points $\mathcal{CM}(E, \Phi, \mathcal{O}_F)$ of type (E, Φ) (see Section 2).

Remark 1.3. The narrow class number 1 assumption in Theorem 1.1 could be removed by working adelically. We have worked in the classical language to emphasize parallels with the original Chowla-Selberg formula.

When $E/\mathbb{Q}$ is a multiquadratic extension (equivalently, $\mathrm{Gal}(E/\mathbb{Q})$ is an elementary abelian 2-group), one can explicitly determine the group of characters X_E (resp. X_F), leading to the following result.

Theorem 1.4. *Let $d_1, \ldots, d_{\ell+1}$ be squarefree, pairwise relatively prime integers with $d_i > 0$ for $i = 1, \ldots, \ell$ and $d_{\ell+1} < 0$ where $\ell = 1$ or 2. Assume that $F = \mathbb{Q}(\sqrt{d_1}, \ldots, \sqrt{d_\ell})$ has narrow class number 1 and let $E = F(\sqrt{d_{\ell+1}})$. Let χ_α (resp. χ_β) be the Kronecker symbol associated to the quadratic field $\mathbb{Q}(\sqrt{\alpha})$ (resp. $\mathbb{Q}(\sqrt{\beta})$), where $\alpha = d_1^{e_1} \cdots d_\ell^{e_\ell} d_{\ell+1}$ (resp. $\beta = d_1^{e_1} \cdots d_\ell^{e_\ell}$) for $\mathbf{e} = (e_1, \ldots, e_\ell) \in \{0,1\}^\ell$. Then*

$$\prod_{[\mathfrak{a}]\in \mathrm{CL}(E)} H(z_{\mathfrak{a}}) = c_1(E, F, 2^\ell) \prod_{\substack{\mathbf{e}\in\{0,1\}^\ell\\ \alpha=d_1^{e_1}\cdots d_\ell^{e_\ell}d_{\ell+1}}} \prod_{k=1}^{c_\alpha} \Gamma\left(\frac{k}{c_\alpha}\right)^{\frac{h_E\chi_\alpha(k)w_\alpha}{4h_\alpha}} \prod_{\substack{\mathbf{e}\in\{0,1\}^\ell\\ \beta=d_1^{e_1}\cdots d_\ell^{e_\ell}\neq 1}} \prod_{k=1}^{c_\beta} \Gamma_2\left(\frac{k}{c_\beta}\right)^{\frac{h_E\chi_\beta(k)}{4h_\beta\log(\varepsilon_\beta)}}.$$

Remark 1.5. The restriction to $\ell = 1$ or 2 in Theorem 1.4 is made for the following reasons. By Fröhlich [F, Theorem 5.6], if F is a totally real abelian field in which at least 5 rational primes ramify, then the class number of F is even. If $\ell \geq 5$, then at least 5 rational primes ramify in $F = \mathbb{Q}(\sqrt{d_1}, \dots, \sqrt{d_\ell})$, hence F cannot have narrow class number 1 (since the class number divides the narrow class number). It is well-known that there exist real quadratic fields of narrow class number 1, and these must be of the form $\mathbb{Q}(\sqrt{2})$ or $\mathbb{Q}(\sqrt{p})$ for a prime $p \equiv 1 \pmod 4$ (see e.g. [CH, Corollary 12.5]). This leaves the possibilities $\ell = 2, 3$ or 4. One can compute many examples of real biquadratic fields with narrow class number 1. We wrote a program in `SAGE` which calculates the narrow class numbers of the real biquadratic fields $F = \mathbb{Q}(\sqrt{p}, \sqrt{q})$ for p and q primes with $2 \leq p < q \leq n$ (see `http://www.math.tamu.edu/~masri/NarrowOne.pdf`). For example, if $n = 30$ there are 6 real biquadratic fields in this list with narrow class number 1, corresponding to the pairs (p, q) given by $\{(2, 5), (2, 13), (2, 29), (5, 13), (5, 17), (17, 29)\}$. On the other hand, for $\ell = 3$ or 4 the class number of $F = \mathbb{Q}(\sqrt{d_1}, \dots, \sqrt{d_\ell})$ can be 1 (see e.g. [Mou]), but we were unable to find any examples with narrow class number 1.

For CM biquadratic fields of class number 1, we have the following result.

Theorem 1.6. *Let $p = 2$ or $p \equiv 1 \pmod 4$ be a prime such that $F = \mathbb{Q}(\sqrt{p})$ has narrow class number 1. Let $d < 0$ be a squarefree integer relatively prime to p such that $E = \mathbb{Q}(\sqrt{p}, \sqrt{d})$ has class number* 1. *Let Δ_p, Δ_d and Δ_{pd} be the discriminants of the quadratic fields $\mathbb{Q}(\sqrt{p})$, $\mathbb{Q}(\sqrt{d})$ and $\mathbb{Q}(\sqrt{pd})$, resp., and assume that Δ_p and Δ_d are relatively prime. Then*

$$H(z_{\mathcal{O}_E}) = \frac{1}{2\sqrt{2\pi|\Delta_d|}} \prod_{k=1}^{|\Delta_d|} \Gamma\left(\frac{k}{|\Delta_d|}\right)^{\frac{\chi_d(k) w_d}{4h_d}} \prod_{k=1}^{|\Delta_{pd}|} \Gamma\left(\frac{k}{|\Delta_{pd}|}\right)^{\frac{\chi_{pd}(k) w_{pd}}{4h_{pd}}} \prod_{k=1}^{\Delta_p} \Gamma_2\left(\frac{k}{\Delta_p}\right)^{\frac{\chi_p(k)}{4\log(\varepsilon_p)}},$$

where

$$z_{\mathcal{O}_E} = \begin{cases} (\sqrt{d}, \sqrt{d}), & d \equiv 2, 3 \pmod 4 \\ \left(\frac{1+\sqrt{d}}{2}, \frac{1+\sqrt{d}}{2}\right), & d \equiv 1 \pmod 4 \end{cases}$$

is a CM point of type (E, Φ) for $\Phi = \{\sigma_1 = \mathrm{id},\ \sigma_2 : \sqrt{p} \mapsto -\sqrt{p}, \sqrt{d} \mapsto \sqrt{d}\}$.

Theorem 1.1 gives a closed form evaluation of the product of CM values $\prod_{[\mathfrak{a}]} H(z_{\mathfrak{a}})$. On the other hand, this product can also be related to Faltings heights of CM abelian varieties, giving a link between the CM values and the arithmetic and geometry of abelian varieties. To explain this relationship, we first recall that the product of CM values $\prod_{[\mathfrak{a}]} G(\tau_{\mathfrak{a}})$ appearing in the classical Chowla-Selberg formula (1.9) can be expressed in terms of the Faltings height of a CM elliptic curve (see e.g. Gross [G] and Silverman [Si]), hence the Chowla-Selberg formula can be reformulated as an identity relating the Faltings height of a CM elliptic curve to the logarithmic derivative of the Dirichlet L–function $L(\chi_d, s)$ at $s = 0$. There is a vast conjectural generalization of this identity due to Colmez [Col] which relates Faltings heights of CM abelian varieties to logarithmic derivatives of Artin L–functions at $s = 0$. Yoshida [Y] made a similar conjecture relating periods of CM abelian varieties (in the sense of Shimura) to logarithmic derivatives of Artin L–functions. See also the work of Anderson [An] for results in this direction.

We now state Colmez's conjecture in the form we will use. Let E be a CM extension of a totally real field F of degree n over $\mathbb{Q}$. Let A be an abelian variety with complex multiplication by E which is defined over $\overline{\mathbb{Q}}$. Let $K \subset \overline{\mathbb{Q}}$ be a number field over which A is defined and let $\omega_A \in H^0(A, \Omega_A^n)$ be a Néron differential. The *Faltings height* of A is defined by (see e.g. [Col, p. 667, (II.2.12.1)])

$$h_{\mathrm{Fal}}(A) := -\frac{1}{[K : \mathbb{Q}]} \left(\sum_{\sigma \in \mathrm{Hom}(K, \overline{\mathbb{Q}})} \frac{1}{2} \log \left(\int_{A^\sigma(\mathbb{C})} |\omega_A^\sigma \wedge \overline{\omega_A^\sigma}| \right) - \sum_{p < \infty} \sum_{\sigma \in \mathrm{Hom}(K, \overline{\mathbb{Q}})} v_p(\omega_A^\sigma) \log(p) \right),$$

where $v_p(\omega_A^\sigma)$ is a certain rational number defined using the p-adic valuation on $\overline{\mathbb{Q}}_p$ (see [Col, p. 659]).

Given a CM type $\Phi \in \Phi(E)$, let A_Φ be a CM abelian variety of type $(\mathcal{O}_E, \Phi)$ defined over $\overline{\mathbb{Q}}$. Colmez [Col, equation (3)] conjectured the following identity for the average of the Faltings heights $h_{\mathrm{Fal}}(A_\Phi)$ of the abelian varieties A_Φ,

$$\frac{1}{2^n}\sum_{\Phi\in\Phi(E)} h_{\mathrm{Fal}}(A_\Phi) = -\frac{1}{2}\left\{\frac{L'(\chi_{E/F},0)}{L(\chi_{E/F},0)} + \frac{1}{2}\log(\mathfrak{f}_{\chi_{E/F}}) + n\log(2\pi)\right\},$$

where $\mathfrak{f}_{\chi_{E/F}}$ is the analytic Artin conductor of the quadratic character $\chi_{E/F}$ (here we have corrected a minor typographical error in the statement of [Col, equation (3)]). When $E/\mathbb{Q}$ is abelian, Colmez [Col, Théorème 5] proved this conjectured identity, up to addition by a possible rational multiple of $\log(2)$. Obus [O] recently completed Colmez's proof by eliminating this possible term. Note that Yang proved the first non-abelian cases of Colmez's conjecture in [Ya].

By combining our results with Colmez's theorem, we will obtain the following result.

Theorem 1.7. *Let $F/\mathbb{Q}$ be a totally real field of degree n and narrow class number 1. Let E/F be a CM extension with $E/\mathbb{Q}$ abelian. Given a CM type $\Phi \in \Phi(E)$, let A_Φ be a CM abelian variety of type $(\mathcal{O}_E, \Phi)$ defined over $\overline{\mathbb{Q}}$ and*

$$\mathcal{CM}(E,\Phi,\mathcal{O}_F) = \{z_{\mathfrak{a}} : [\mathfrak{a}] \in \mathrm{CL}(E)\}$$

be a set of CM points of type (E, Φ). Then

$$\prod_{[\mathfrak{a}]\in\mathrm{CL}(E)} H(z_{\mathfrak{a}}) = c_2(E,F,n)\prod_{\substack{\chi\in X_F\\ \chi\neq 1}}\prod_{k=1}^{c_\chi}\Gamma_2\left(\frac{k}{c_\chi}\right)^{\frac{h_E\tau(\chi)\overline{\chi}(k)}{2c_\chi L(\chi,1)}}\prod_{\Phi\in\Phi(E)}\exp(h_{\mathrm{Fal}}(A_\Phi))^{-\frac{h_E}{2^n}},$$

where

$$c_2(E,F,n) := \left(\sqrt{\frac{d_E}{\mathfrak{f}_{\chi_{E/F}}}}\frac{1}{2^{2n+1}\pi^{n+1}d_F^2}\right)^{\frac{h_E}{2}}.$$

On the other hand, we will use Colmez's theorem to evaluate products of exponentials of Faltings heights in terms of values of $\Gamma(s)$ at rational numbers.

Proposition 1.8. *Let $d_1, \dots, d_{\ell+1}$ be squarefree, pairwise relatively prime integers such that $d_i > 0$ for $i = 1, \dots, \ell$ and $d_{\ell+1} < 0$ where $\ell \in \mathbb{Z}^+$. Let $F = \mathbb{Q}(\sqrt{d_1}, \dots, \sqrt{d_\ell})$ and $E = F(\sqrt{d_{\ell+1}})$. Let χ_α be the Kronecker symbol associated to the quadratic field $\mathbb{Q}(\sqrt{\alpha})$ where $\alpha = d_1^{e_1}\cdots d_\ell^{e_\ell}d_{\ell+1}$ for $\mathbf{e} = (e_1, \dots, e_\ell) \in \{0,1\}^\ell$. Given a CM type $\Phi \in \Phi(E)$, let A_Φ be a CM abelian variety of type $(\mathcal{O}_E, \Phi)$ defined over $\overline{\mathbb{Q}}$. Then*

$$\prod_{\Phi\in\Phi(E)}\exp(h_{\mathrm{Fal}}(A_\Phi)) = c_3(E,F,\ell)\prod_{\substack{\mathbf{e}\in\{0,1\}^\ell\\ \alpha=d_1^{e_1}\cdots d_\ell^{e_\ell}d_{\ell+1}}}\prod_{k=1}^{c_\alpha}\Gamma\left(\frac{k}{c_\alpha}\right)^{\frac{-2^{2^\ell-2}\chi_\alpha(k)w_\alpha}{h_\alpha}},$$

where

$$c_3(E,F,\ell) := \left(\frac{(2\pi)^{2^\ell}d_F\sqrt{\mathfrak{f}_{\chi_{E/F}}}}{d_E}\right)^{-2^{2^\ell-1}}.$$

Remark 1.9. Proposition 1.8 should be compared with [Col, Remarque on p. 680] and [BMM-B].

The formula in Proposition 1.8 allows us to study some arithmetic properties of the products

$$\prod_{\Phi \in \Phi(E)} \exp(h_{\mathrm{Fal}}(A_\Phi))$$

using the Lang-Rohrlich conjecture (see e.g. [L, Appendix to Section 2, p. 66]). Roughly speaking, the Lang-Rohrlich conjecture states that all polynomial algebraic relations among the special Γ-values $\{\Gamma(s) : s \in \mathbb{Q}\backslash\mathbb{Z}_{\leq 0}\}$ and $2\pi i$ with coefficients in $\overline{\mathbb{Q}}$ are "explained" by the standard functional equations. One can formulate this conjecture as a converse of the Koblitz-Ogus criterion for an element of the subgroup of $\mathbb{C}^\times$ generated by the special Γ-values and $2\pi i$ to belong to $\overline{\mathbb{Q}}^\times$ (see the Appendix to [De] and Section 12). For a more detailed discussion of the Lang-Rohrlich conjecture, including its various formulations and known results in this direction, see the introduction to Anderson, Brownawell and Papanikolas [ABP].

We will prove the following result, which should be compared with the classical result of Euler

$$\zeta(2n) \in \pi^{2n}\mathbb{Q}, \quad n = 1, 2, \ldots$$

concerning values of the Riemann zeta function $\zeta(s)$ at positive even integers.

Theorem 1.10. *Let $d_1, \ldots, d_{\ell+1}$ be squarefree, pairwise relatively prime integers such that $d_i > 0$ for $i = 1, \ldots, \ell$ and $d_{\ell+1} < 0$ where $\ell \in \mathbb{Z}^+$. Let $F = \mathbb{Q}(\sqrt{d_1}, \ldots, \sqrt{d_\ell})$ and $E = F(\sqrt{d_{\ell+1}})$. Given a CM type $\Phi \in \Phi(E)$, let A_Φ be a CM abelian variety of type $(\mathcal{O}_E, \Phi)$ defined over $\overline{\mathbb{Q}}$. Then assuming the Lang-Rohrlich conjecture,*

$$\prod_{\Phi \in \Phi(E)} \exp(h_{\mathrm{Fal}}(A_\Phi)) \notin \pi^k \overline{\mathbb{Q}}$$

for any $k \in \mathbb{Q}$.

1.4. **Connection to some existing work.** We conclude the introduction by discussing the connection between our results and some existing work. A version of the Chowla-Selberg formula for CM fields was given by Moreno [Mo] over 30 years ago. The foundation for such a generalization was laid by Asai [A] in the late 1960's, who established a Kronecker limit formula for Eisenstein series associated to any number field of class number 1. Following Weil's [W, Chapter IX] beautiful exposition of the classical Chowla-Selberg formula (which involves a renormalized Kronecker limit formula for Eisenstein series over $\mathbb{Q}$), Moreno obtained an expression relating values of a Hilbert modular function at special points on a Hilbert-Blumenthal variety to the logarithmic derivative of $L(\chi_{E/F}, s)$ at $s = 0$. Moreno then used Shintani's [Sh1, Sh2] remarkable work on special values of L–functions to express $L'(\chi_{E/F}, 0)$ in terms of certain Barnes-type multiple gamma functions (formulas of this type resulting from Shintani's work can be viewed as "higher" analogs of Lerch's identity). Putting things together, he obtained a version of the Chowla-Selberg formula for CM fields (see [Mo, Main Theorem, p. 242]). The starting point of this paper was that it should be possible to give a much more explicit version of the Chowla-Selberg formula for abelian CM fields. The initial structure of the proof is similar to that of Moreno's, namely to arrive at a version of the identity (1.10), though there are important differences. For example, we identify the CM zero-cycles along which we evaluate the Hilbert modular Eisenstein series, which allows us to give explicit examples of our formula (see Section 2) and paves the way to relate the CM values of $H(z)$ to the arithmetic and geometry of CM abelian varieties via Colmez's conjecture.

Acknowledgments. We would like to thank Matt Papanikolas for helpful discussions regarding this work. We would also like to thank the referee for several helpful remarks. The second author was supported by the NSF grant DMS-1162535 during the preparation of this work.

2. Examples

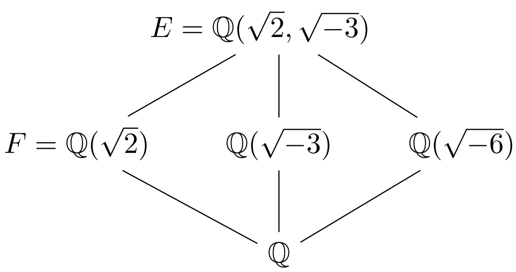
In this section we give some explicit examples of the Chowla-Selberg formula for abelian CM fields. Recall that the function $H : \mathbb{H}^n \to \mathbb{R}^+$ appearing in these examples is a $\mathrm{SL}_2(\mathcal{O}_F)$-invariant function analogous to $G(z) := \sqrt{\mathrm{Im}(z)}|\eta(z)|^2$ which arises from a renormalized Kronecker limit formula for the non-holomorphic Hilbert modular Eisenstein series. See (3.8) for the definition of $H(z)$. For background and notation regarding CM points, see Section 4.

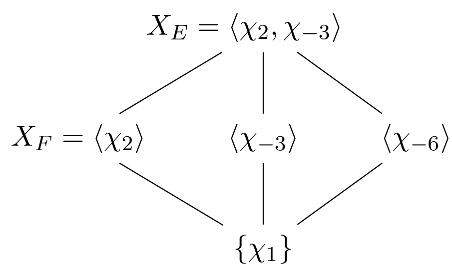
Example 2.1 (Theorem 1.6, $d_1 = 2$ and $d_2 = -3$)**.** Let $E = \mathbb{Q}(\sqrt{2}, \sqrt{-3})$ and $F = \mathbb{Q}(\sqrt{2})$. Then E has class number 1 and F has narrow class number 1. Moreover, $\Delta_2 = 8, \Delta_{-3} = -3$ and $\Delta_{-6} = -24$, so that Δ_2 and Δ_{-3} are relatively prime. The hypotheses of Theorem 1.6 are satisfied, so it remains to determine the quantities in the identity stated in Theorem 1.6.

Since $-3 \equiv 1 \mod 4$, the CM point of type (E, Φ) corresponding to the class $[\mathcal{O}_E]$ is given by

$$z_{\mathcal{O}_E} = \left(\frac{1+\sqrt{-3}}{2}, \frac{1+\sqrt{-3}}{2} \right).$$

The groups of characters associated to E and F are $X_E = \{\chi_1, \chi_{-3}, \chi_2, \chi_{-6}\}$ and $X_F = \{\chi_1, \chi_2\}$, resp., hence $X_E \setminus X_F = \{\chi_{-3}, \chi_{-6}\}$. We have the following correspondence between subfields and associated character groups:





The characters $\chi_2 = \left(\frac{8}{\cdot}\right)$, $\chi_{-6} = \left(\frac{-24}{\cdot}\right)$ and $\chi_{-3} = \left(\frac{-3}{\cdot}\right)$ have conductors 8, 24 and 3, resp. (note the character χ_2 generates X_F and the characters χ_{-3} and χ_2 generate X_E). The following tables give the values of these characters:

Values of $\chi_2 = \left(\frac{8}{\cdot}\right)$				
k	1	3	5	7
$\chi_2(k)$	1	-1	-1	1

Values of $\chi_{-6} = \left(\frac{-24}{\cdot}\right)$								
k	1	5	7	11	13	17	19	23
$\chi_{-6}(k)$	1	1	1	1	-1	-1	-1	-1

Values of $\chi_{-3} = \left(\frac{-3}{\cdot}\right)$		
k	1	2
$\chi_{-3}(k)$	1	-1

The fundamental unit of F is $\varepsilon_2 = 1 + \sqrt{2}$, and we have $h_{-3} = 1, h_{-6} = 2, w_{-3} = 6$ and $w_{-6} = 2$. Substituting these quantities in Theorem 1.6 yields

$$H(z_{\mathcal{O}_E}) = \frac{1}{2\sqrt{6}\pi} \prod_{k=1}^{3} \Gamma\left(\frac{k}{3}\right)^{\frac{3\chi_{-3}(k)}{2}} \prod_{k=1}^{24} \Gamma\left(\frac{k}{24}\right)^{\frac{\chi_{-6}(k)}{4}} \prod_{k=1}^{8} \Gamma_2\left(\frac{k}{8}\right)^{\frac{\chi_2(k)}{4\log(1+\sqrt{2})}}.$$

After expanding each product on the right hand side, we get

$$H\left(\frac{1+\sqrt{-3}}{2}, \frac{1+\sqrt{-3}}{2}\right) = \frac{1}{2\sqrt{6}\pi}\left(\frac{\Gamma\left(\frac{1}{3}\right)}{\Gamma\left(\frac{2}{3}\right)}\right)^{3/2}\left(\frac{\Gamma\left(\frac{1}{24}\right)\Gamma\left(\frac{5}{24}\right)\Gamma\left(\frac{7}{24}\right)\Gamma\left(\frac{11}{24}\right)}{\Gamma\left(\frac{13}{24}\right)\Gamma\left(\frac{17}{24}\right)\Gamma\left(\frac{19}{24}\right)\Gamma\left(\frac{23}{24}\right)}\right)^{1/4}\left(\frac{\Gamma_2\left(\frac{1}{8}\right)\Gamma_2\left(\frac{7}{8}\right)}{\Gamma_2\left(\frac{3}{8}\right)\Gamma_2\left(\frac{5}{8}\right)}\right)^{\frac{1}{4\log(1+\sqrt{2})}}.$$

Example 2.2 (Theorem 1.4, $d_1 = 2$ and $d_2 = -5$)**.** Let $E = \mathbb{Q}(\sqrt{2}, \sqrt{-5})$ and $F = \mathbb{Q}(\sqrt{2})$. Then E has class number 2 and F has narrow class number 1. Moreover, $d_1 = 2$ and $d_2 = -5$ are squarefree and relatively prime. The hypotheses of Theorem 1.4 are satisfied, so it remains to determine the quantities in the identity stated in Theorem 1.4.

The four embeddings of E are determined by

$$\begin{aligned}
\sigma_1 &: \sqrt{2} \longmapsto \sqrt{2}, \quad && \sqrt{-5} \longmapsto \sqrt{-5}\\
\sigma_2 &: \sqrt{2} \longmapsto -\sqrt{2}, \quad && \sqrt{-5} \longmapsto \sqrt{-5}\\
\sigma_3 &: \sqrt{2} \longmapsto \sqrt{2}, \quad && \sqrt{-5} \longmapsto -\sqrt{-5}\\
\sigma_4 &: \sqrt{2} \longmapsto -\sqrt{2}, \quad && \sqrt{-5} \longmapsto -\sqrt{-5}.
\end{aligned}$$

Fix the choice of CM type $\Phi = \{\sigma_1, \sigma_2\}$. The class group of E is given by $\mathrm{CL}(E) = \{[\mathcal{O}_E], [\mathfrak{a}]\}$ where

$$\begin{aligned}
[\mathcal{O}_E] &= [\mathcal{O}_F(10-\sqrt{2}) + \mathcal{O}_F(\sqrt{-5}+18\sqrt{2}-1)],\\
[\mathfrak{a}] &= [\mathcal{O}_F 2 + \mathcal{O}_F(\sqrt{-5}-\sqrt{2}+1)].
\end{aligned}$$

Then

$$z_{\mathcal{O}_E} = \frac{\sqrt{-5}+18\sqrt{2}-1}{10-\sqrt{2}} \qquad \text{and} \qquad z_{\mathfrak{a}} = \frac{\sqrt{-5}-\sqrt{2}-1}{2}$$

are CM points of type (E, Φ) corresponding to the classes $[\mathcal{O}_F]$ and $[\mathfrak{a}]$ resp., since

$$\Phi(z_{\mathcal{O}_E}) = \left(\frac{\sqrt{-5}+18\sqrt{2}-1}{10-\sqrt{2}}, \frac{\sqrt{-5}-18\sqrt{2}-1}{10+\sqrt{2}}\right) \in E^\times \cap \mathbb{H}^2$$

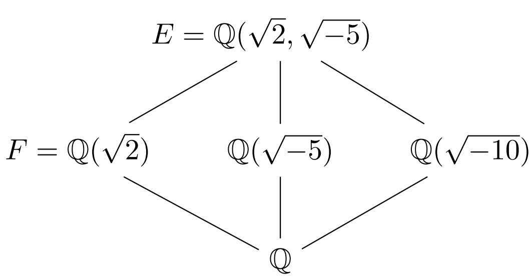
and

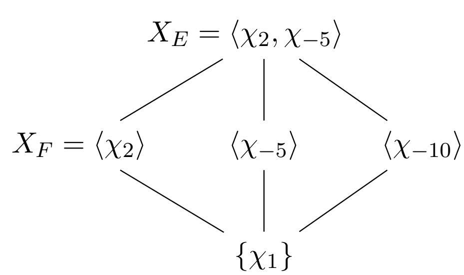
$$\Phi(z_{\mathfrak{a}}) = \left(\frac{\sqrt{-5}-\sqrt{2}-1}{2}, \frac{\sqrt{-5}+\sqrt{2}-1}{2}\right) \in E^\times \cap \mathbb{H}^2.$$

The absolute values of the discriminants of E and F are $d_E = 6400$ and $d_F = 8$, resp., hence the constant

$$c_1(E, F, 2) = \frac{8}{2^3\pi\sqrt{6400}} = \frac{1}{80\pi}.$$

The groups of characters associated to E and F are $X_E = \{\chi_1, \chi_2, \chi_{-5}, \chi_{-10}\}$ and $X_F = \{\chi_1, \chi_2\}$, resp., hence $X_E \setminus X_F = \{\chi_{-5}, \chi_{-10}\}$. We have the following correspondence between subfields and associated character groups:





The characters $\chi_2 = \left(\frac{8}{\cdot}\right)$, $\chi_{-5} = \left(\frac{-20}{\cdot}\right)$ and $\chi_{-10} = \left(\frac{-40}{\cdot}\right)$ have conductors 8, 20 and 40, resp. (note the character χ_2 generates X_F and the characters χ_2 and χ_{-5} generate X_E). The following tables give the values of these characters:

Values of $\chi_2 = \left(\frac{8}{\cdot}\right)$				
k	1	3	5	7
$\chi_2(k)$	1	−1	−1	1

Values of $\chi_{-5} = \left(\frac{-20}{\cdot}\right)$								
k	1	3	7	9	11	13	17	19
$\chi_{-5}(k)$	1	1	1	1	−1	−1	−1	−1

Values of $\chi_{-10} = \left(\frac{-40}{\cdot}\right)$																
k	1	3	7	9	11	13	17	19	21	23	27	29	31	33	37	39
$\chi_{-10}(k)$	1	−1	1	1	1	1	−1	1	−1	1	−1	−1	−1	−1	1	−1

The fundamental unit of F is $\varepsilon_2 = 1 + \sqrt{2}$, and we have $h_2 = 1$, $h_{-5} = 2$, $h_{-10} = 2$, $w_2 = 2$, $w_{-5} = 2$ and $w_{-10} = 2$.

Substituting the preceding quantities in Theorem 1.4 yields

$$H(z_{\mathcal{O}_E})H(z_{\mathfrak{a}}) = \frac{1}{80\pi} \prod_{k=1}^{20} \Gamma\left(\frac{k}{20}\right)^{\frac{\chi_{-5}(k)}{2}} \prod_{k=1}^{40} \Gamma\left(\frac{k}{40}\right)^{\frac{\chi_{-10}(k)}{2}} \prod_{k=1}^{8} \Gamma_2\left(\frac{k}{8}\right)^{\frac{\chi_2(k)}{2\log(1+\sqrt{2})}}.$$

After expanding each product on the right hand side, we get

$$\begin{aligned} &H\left(\frac{\sqrt{-5}+18\sqrt{2}-1}{10-\sqrt{2}}, \frac{\sqrt{-5}-18\sqrt{2}-1}{10+\sqrt{2}}\right) H\left(\frac{\sqrt{-5}-\sqrt{2}-1}{2}, \frac{\sqrt{-5}+\sqrt{2}-1}{2}\right) \\ &= \frac{1}{80\pi}\left(\frac{\Gamma(\frac{1}{20})\Gamma(\frac{3}{20})\Gamma(\frac{7}{20})\Gamma(\frac{9}{20})}{\Gamma(\frac{11}{20})\Gamma(\frac{13}{20})\Gamma(\frac{17}{20})\Gamma(\frac{19}{20})}\right)^{1/2} \left(\frac{\Gamma(\frac{1}{40})\Gamma(\frac{7}{40})\Gamma(\frac{9}{40})\Gamma(\frac{11}{40})\Gamma(\frac{13}{40})\Gamma(\frac{19}{40})\Gamma(\frac{23}{40})\Gamma(\frac{37}{40})}{\Gamma(\frac{3}{40})\Gamma(\frac{17}{40})\Gamma(\frac{21}{40})\Gamma(\frac{27}{40})\Gamma(\frac{29}{40})\Gamma(\frac{31}{40})\Gamma(\frac{33}{40})\Gamma(\frac{39}{40})}\right)^{1/2} \\ &\quad \times \left(\frac{\Gamma_2(\frac{1}{8})\Gamma_2(\frac{7}{8})}{\Gamma_2(\frac{3}{8})\Gamma_2(\frac{5}{8})}\right)^{\frac{1}{2\log(1+\sqrt{2})}}. \end{aligned}$$

Example 2.3 (Theorem 1.1, $E = \mathbb{Q}(\zeta_5)$ and $F = \mathbb{Q}(\sqrt{5})$)**.** Let $E = \mathbb{Q}(\zeta_5)$ and $F = \mathbb{Q}(\sqrt{5})$. Then E is a CM extension of the real quadratic field F with $E/\mathbb{Q}$ abelian (a cyclic quartic extension). Moreover, E has class number 1 and F has narrow class number 1. The hypotheses of Theorem 1.1 are satisfied, so it remains to determine the quantities in the identity stated in Theorem 1.1.

The four embeddings of E are determined by $\sigma_i(\zeta_5) = \zeta_5^i$ for $i = 1, \ldots, 4$. Fix the choice of CM type $\Phi = \{\sigma_1, \sigma_2\}$ for E. We have $\mathcal{O}_E = \mathcal{O}_F + \mathcal{O}_F\zeta_5$, thus $z_{\mathcal{O}_E} = \zeta_5$ is a CM point of type (E, Φ) since $\Phi(z_{\mathcal{O}_E}) = (\zeta_5, \zeta_5^2) \in E^\times \cap \mathbb{H}^2$.

The absolute values of the discriminants are $d_E = 125$ and $d_F = 5$, resp., hence the constant

$$c_1(E, F, 2) = \left(\frac{1}{8\pi\sqrt{5}}\right)^{1/2}.$$

Since $E = \mathbb{Q}(\zeta_5)$ is cyclotomic, we have $X_E = \widehat{(\mathbb{Z}/5\mathbb{Z})^\times}$. The following table gives the group of Dirichlet characters modulo 5:

Dirichlet characters modulo 5				
	1	2	3	4
χ_1	1	1	1	1
χ	1	i	$-i$	-1
$\chi^2 = \chi_5 = \left(\frac{5}{\cdot}\right)$	1	-1	-1	1
$\chi^3 = \overline{\chi}$	1	$-i$	i	-1

We have the following correspondence between subfields and associated character groups:

$$\begin{array}{ccc} E = \mathbb{Q}(\zeta_5) & \qquad & X_E = \langle \chi \rangle \\ | & & | \\ F = \mathbb{Q}(\sqrt{5}) & & X_F = \langle \chi_5 \rangle \\ | & & | \\ \mathbb{Q} & & \{\chi_1\} \end{array}$$

It follows that $X_F = \{\chi_1, \chi^2\} = \{\chi_1, \chi_5\}$ and $X_E \setminus X_F = \{\chi, \chi^3\} = \{\chi, \overline{\chi}\}$.

The L–values corresponding to the characters $\chi, \overline{\chi}$ are given in terms of generalized Bernoulli numbers by

$$L(\chi, 0) = -B_1(\chi) = \frac{3}{5} + \frac{1}{5}i \quad \text{and} \quad L(\overline{\chi}, 0) = -B_1(\overline{\chi}) = \frac{3}{5} - \frac{1}{5}i.$$

Moreover, by the class number formula we have

$$L(\chi_5, 1) = \frac{2\log(\frac{1+\sqrt{5}}{2})}{\sqrt{5}},$$

the Gauss sum is evaluated as $\tau(\chi_5) = \sqrt{5}$, and the fundamental unit of F is $\varepsilon_5 = \frac{1+\sqrt{5}}{2}$.

Substituting the preceding quantities in Theorem 1.1 yields

$$H(z_{\mathcal{O}_E}) = \left(\frac{1}{8\pi\sqrt{5}}\right)^{1/2} \prod_{k=1}^{5} \Gamma\left(\frac{k}{5}\right)^{\frac{\chi(k)}{2\left(\frac{3}{5}+\frac{1}{5}i\right)}} \prod_{k=1}^{5} \Gamma\left(\frac{k}{5}\right)^{\frac{\overline{\chi}(k)}{2\left(\frac{3}{5}-\frac{1}{5}i\right)}} \prod_{k=1}^{5} \Gamma_2\left(\frac{k}{5}\right)^{\frac{\chi_5(k)}{4\log\left(\frac{1+\sqrt{5}}{2}\right)}}.$$

After expanding each product on the right hand side, we get

$$H(\zeta_5, \zeta_5^2) = \left(\frac{1}{8\pi\sqrt{5}}\right)^{1/2} \left(\frac{\Gamma\left(\frac{1}{5}\right)}{\Gamma\left(\frac{4}{5}\right)}\right)^{3/2} \left(\frac{\Gamma\left(\frac{2}{5}\right)}{\Gamma\left(\frac{3}{5}\right)}\right)^{1/2} \left(\frac{\Gamma_2\left(\frac{1}{5}\right)\Gamma_2\left(\frac{4}{5}\right)}{\Gamma_2\left(\frac{2}{5}\right)\Gamma_2\left(\frac{3}{5}\right)}\right)^{\frac{1}{4\log\left(\frac{1+\sqrt{5}}{2}\right)}}.$$

3. Hilbert modular Eisenstein Series

In this section we establish a renormalized Kronecker limit formula for the non-holomorphic Hilbert modular Eisenstein series. Moreno stated such a formula in [Mo, Section 3.1], and gave a very brief explanation as to how it is derived from a Fourier expansion of Asai [A] for the Eisenstein series. Here we give a similar formula using a slightly different form of the Fourier expansion (the Fourier expansion we use for the Hilbert modular Eisenstein series goes back to Hecke).

Let F be a totally real number field of degree n over $\mathbb{Q}$ with embeddings $\tau_1, \ldots, \tau_n$. Let

$$z = x + iy = (z_1, \ldots, z_n) \in \mathbb{H}^n$$

where $\mathbb{H}$ denotes the complex upper half-plane. Let $\mathcal{O}_F$ be the ring of integers of F and $\mathrm{SL}_2(\mathcal{O}_F)$ be the Hilbert modular group. Then $\mathrm{SL}_2(\mathcal{O}_F)$ acts componentwise on $\mathbb{H}^n$ by linear fractional transformations,

$$Mz = (\tau_1(M)z_1, \ldots, \tau_n(M)z_n), \quad M = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \mathrm{SL}_2(\mathcal{O}_F)$$

where

$$\tau_j(M) = \begin{pmatrix} \tau_j(\alpha) & \tau_j(\beta) \\ \tau_j(\gamma) & \tau_j(\delta) \end{pmatrix}.$$

Let

$$N(y(z)) := \prod_{j=1}^{n} \mathrm{Im}(z_j) = \prod_{j=1}^{n} y_j$$

denote the product of the imaginary parts of the components of $z \in \mathbb{H}^n$. Define the non-holomorphic Hilbert modular Eisenstein series

$$E(z,s) := \sum_{M \in \Gamma_\infty \backslash \mathrm{SL}_2(\mathcal{O}_F)} N(y(Mz))^s, \quad z \in \mathbb{H}^n, \quad \mathrm{Re}(s) > 1$$

where

$$\Gamma_\infty = \left\{ \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \in \mathrm{SL}_2(\mathcal{O}_F) \right\}.$$

Furthermore, let

$$N(a+bz) := \prod_{j=1}^{n} (\sigma_j(a) + \sigma_j(b) z_j)$$

for $(a,b) \in \mathcal{O}_F \times \mathcal{O}_F$ and define the Epstein zeta function

$$Z(z,s) := \sideset{}{'}\sum_{(a,b) \in \mathcal{O}_F \times \mathcal{O}_F / \mathcal{O}_F^\times} \frac{N(y(z))^s}{|N(a+bz)|^{2s}}, \quad z \in \mathbb{H}^n, \quad \mathrm{Re}(s) > 1$$

where the sum is over a complete set of nonzero, nonassociated representatives of $\mathcal{O}_F \times \mathcal{O}_F$ (recall that (a,b) and (a',b') are said to be *associated* if there exists a unit $\epsilon \in \mathcal{O}_F^\times$ such that $(a,b) = (\epsilon a', \epsilon b')$). One has the identity

$$Z(z,s) = \zeta_F(2s) E(z,s), \tag{3.1}$$

where $\zeta_F(s)$ is the Dedekind zeta function of F.

Define the completed Eisenstein series

$$E^*(z,s) := \zeta_F^*(2s) E(z,s) \tag{3.2}$$

where

$$\zeta_F^*(s) := d_F^{s/2} \pi^{-ns/2} \Gamma(s/2)^n \zeta_F(s),$$

is the completed Dedekind zeta function of F.

From [vG, Proposition 6.9], equation (3.2), and the shift $s \mapsto (s+1)/2$, we obtain the renormalized Fourier expansion

$$\begin{aligned} E\left(z, \frac{s+1}{2}\right) &= N(y)^{\frac{s+1}{2}} + \frac{\zeta_F^*(s)}{\zeta_F^*(s+1)} N(y)^{\frac{1-s}{2}} \\ &\quad + \frac{2^n N(y)^{1/2}}{\zeta_F^*(s+1)} \sum_{\substack{\mu \in \partial_F^{-1}/\mathcal{O}_F^\times \\ \mu \neq 0}} N_{F/\mathbb{Q}}((\mu)\partial_F)^{\frac{s}{2}} \sigma_{-s}((\mu)\partial_F) \prod_{j=1}^{n} K_{\frac{s}{2}}(2\pi |\mu^{(j)}| y_j) e^{2\pi i \, \mathrm{Tr}(\mu x)}, \end{aligned} \tag{3.3}$$

where ∂_F is the different of F,

$$\sigma_\nu(\mathfrak{a}) := \sum_{\mathfrak{b}|\mathfrak{a}} N_{F/\mathbb{Q}}(\mathfrak{b})^\nu$$

is the divisor function,

$$\mathrm{Tr}(\mu x) := \sum_{j=1}^{n} \mu^{(j)} x_j, \quad \mu^{(j)} := \tau_j(\mu)$$

is the trace and

$$K_s(t) := \int_0^\infty e^{-t\cosh x} \cosh{(sx)} dx$$

is the K-Bessel function of order s.

Let $A(s), B(s)$ and $C(s)$ denote the first, second, and third terms on the right hand side of (3.3), resp. We compute the first two terms in the Taylor expansion of $E\left(z, \frac{s+1}{2}\right)$ at $s = -1$ by doing this for each of the functions $A(s), B(s)$ and $C(s)$, in turn.

First, observe that

$$A(s) = 1 + \log N(y)^{1/2}(s+1) + O((s+1)^2).$$

Second, we calculate the Taylor expansion

$$B(s) = B(-1) + B'(-1)(s+1) + O(s+1)^2.$$

Since

$$\frac{1}{\zeta_F^*(s)} = d_F^{-s/2} \left(\frac{\pi^{s/2}}{\Gamma(s/2)}\right)^n \frac{1}{\zeta_F(s)}$$

and $\zeta_F(s)$ has a simple pole at $s = 1$, the function $1/\zeta_F^*(s)$ has a simple zero at $s = 1$. Using the functional equation $\zeta_F^*(s) = \zeta_F^*(1-s)$, it follows that

(∗) $1/\zeta_F^*(s)$ has a simple zero at $s = 0$.

Now, by (∗) we have

$$B(-1) = \frac{\zeta_F^*(-1)}{\zeta_F^*(0)} N(y) = 0.$$

Moreover, an application of the product and quotient rules along with two applications of (∗) yields

$$B'(s) = -N(y)^{\frac{1-s}{2}} \zeta_F^*(s) \left(\frac{\frac{d}{ds}\zeta_F^*(s+1)}{\zeta_F^*(s+1)^2}\right) + O(s+1),$$

so that

$$B'(-1) = -N(y)\zeta_F^*(-1)\frac{(\zeta_F^*)'(0)}{\zeta_F^*(0)^2}.$$

A calculation using the Laurent expansion

$$\zeta_F^*(s+1) = \frac{r_F}{s+1} + O(s+1)$$

yields

$$-\frac{\frac{d}{ds}\zeta_F^*(s+1)}{\zeta_F^*(s+1)^2} = \frac{r_F + O(s+1)^2}{\{r_F + O(s+1)\}^2}, \tag{3.4}$$

where r_F is the residue of $\zeta_F^*(s+1)$ at $s = -1$. Hence

$$B'(-1) = \frac{N(y)\zeta_F^*(-1)}{r_F}.$$

Third, we calculate the Taylor expansion

$$C(s) = C(-1) + C'(-1)(s+1) + O(s+1)^2.$$

For convenience, we write

$$C(s) = 2^n N(y)^{1/2} \sum_{\substack{\mu \in \partial_F^{-1}/\mathcal{O}_F^\times \\ \mu \neq 0}} D_\mu(s) e^{2\pi i \operatorname{Tr}(\mu x)},$$

where

$$D_\mu(s) := \frac{N_{F/\mathbb{Q}}((\mu)\partial_F)^{\frac{s}{2}}}{\zeta_F^*(s+1)} \sigma_{-s}((\mu)\partial_F) \prod_{j=1}^{n} K_{\frac{s}{2}}(2\pi|\mu^{(j)}|y_j).$$

By $(*)$ we have $D_\mu(-1) = 0$, thus $C(-1) = 0$.

To compute $C'(-1)$, it suffices to compute $D'_\mu(-1)$. Using the product rule, two applications of $(*)$, and (3.4) we obtain

$$D'_\mu(-1) = \frac{N_{F/\mathbb{Q}}((\mu)\partial_F)^{-\frac{1}{2}}}{r_F} \sigma_1((\mu)\partial_F) \prod_{j=1}^{n} K_{-\frac{1}{2}}(2\pi|\mu^{(j)}|y_j).$$

A calculation using the identities $K_{-s}(t) = K_s(t)$ and $K_{\frac{1}{2}}(t) = \sqrt{\frac{\pi}{2}} e^{-t} t^{-\frac{1}{2}}$ for $t > 0$ gives

$$D'_\mu(-1) = \frac{N_{F/\mathbb{Q}}((\mu)\partial_F)^{-\frac{1}{2}}}{r_F} \sigma_1((\mu)\partial_F) \prod_{j=1}^{n} \sqrt{\frac{\pi}{2}} e^{-2\pi|\mu^{(j)}|y_j} (2\pi|\mu^{(j)}|y_j)^{-\frac{1}{2}}. \tag{3.5}$$

Note also that

$$\left\{ \prod_{j=1}^{n} \sqrt{\frac{\pi}{2}} e^{-2\pi|\mu^{(j)}|y_j} (2\pi|\mu^{(j)}|y_j)^{-\frac{1}{2}} \right\} e^{2\pi i \operatorname{Tr}(\mu x)} = 2^{-n} |N_{F/\mathbb{Q}}(\mu)|^{-\frac{1}{2}} N(y)^{-\frac{1}{2}} e^{2\pi i T(\mu,z)}, \tag{3.6}$$

where

$$T(\mu, z) := \operatorname{Tr}(\mu x) + i \sum_{j=1}^{n} |\mu^{(j)}| y_j.$$

Then using (3.5) and (3.6), we get

$$C'(-1) = \sum_{\substack{\mu \in \partial_F^{-1}/\mathcal{O}_F^\times \\ \mu \neq 0}} \frac{N_{F/\mathbb{Q}}((\mu)\partial_F)^{-\frac{1}{2}}}{r_F} \sigma_1((\mu)\partial_F) |N_{F/\mathbb{Q}}(\mu)|^{-\frac{1}{2}} e^{2\pi i T(\mu,z)}.$$

Finally, by combining the Taylor expansions for $A(s), B(s)$ and $C(s)$, we obtain the following result.

Proposition 3.1. *We have*

$$E\left(z, \frac{s+1}{2}\right) = 1 + \log\left(H(z)\right)(s+1) + O((s+1)^2), \tag{3.7}$$

where

$$H(z) := \sqrt{N(y)} \phi(z) \tag{3.8}$$

and

$$\log(\phi(z)) := \frac{\zeta_F^*(-1)N(y)}{r_F} + \sum_{\substack{\mu\in\partial_F^{-1}/\mathcal{O}_F^\times \\ \mu\neq 0}} \frac{N_{F/\mathbb{Q}}((\mu)\partial_F)^{-\frac{1}{2}}}{r_F}\sigma_1((\mu)\partial_F)|N_{F/\mathbb{Q}}(\mu)|^{-\frac{1}{2}}e^{2\pi i T(\mu,z)}.$$

Remark 3.2. Using (3.7) and the automorphy of $E(z,s)$, we have $H(Mz) = H(z)$ for all $M = \begin{pmatrix}\alpha & \beta \\ \gamma & \delta\end{pmatrix} \in \mathrm{SL}_2(\mathcal{O}_F)$. Then a straightforward calculation yields the transformation formula

$$\phi(Mz) = |N(\gamma z+\delta)|\,\phi(z).$$

4. CM zero-cycles on Hilbert modular varieties

In this section we summarize some facts we will need regarding CM zero-cycles on Hilbert modular varieties. For more details, see [BY, Section 3]. Let F be a totally real number field of degree n over $\mathbb{Q}$ with embeddings $\tau_1,\ldots,\tau_n$, and assume that F has *narrow class number one*. The quotient $X(\mathcal{O}_F) = \mathrm{SL}_2(\mathcal{O}_F)\backslash\mathbb{H}^n$ is the (open) Hilbert modular variety associated to $\mathcal{O}_F$. The variety $X(\mathcal{O}_F)$ parametrizes isomorphism classes of principally polarized abelian varieties (A,i) with real multiplication $i:\mathcal{O}_F \hookrightarrow \mathrm{End}(A)$.

Let E be a CM extension of F and $\Phi = (\sigma_1,\ldots,\sigma_n)$ be a CM type for E. A point $z = (A,i) \in X(\mathcal{O}_F)$ is a *CM point* of type (E,Φ) if one of the following equivalent conditions holds:

(1) As a point $z \in \mathbb{H}^n$, there is a point $\tau \in E$ such that

$$\Phi(\tau) = (\sigma_1(\tau),\ldots,\sigma_n(\tau)) = z$$

and

$$\Lambda_\tau = \mathcal{O}_F + \mathcal{O}_F\tau$$

is a fractional ideal of E.

(2) There exists a pair (A,i') that is a CM abelian variety of type (E,Φ) with complex multiplication $i' : \mathcal{O}_E \hookrightarrow \mathrm{End}(A)$ such that $i = i'|_{\mathcal{O}_F}$.

By [BY, Lemma 3.2] and the narrow class number one assumption, there is a bijection between the ideal class group $\mathrm{CL}(E)$ and the CM points of type (E,Φ) defined as follows: given an ideal class $C \in \mathrm{CL}(E)$, there exists a fractional ideal $\mathfrak{a} \in C^{-1}$ and $\alpha,\beta \in E^\times$ such that

$$\mathfrak{a} = \mathcal{O}_F\alpha + \mathcal{O}_F\beta \tag{4.1}$$

and

$$z = \frac{\beta}{\alpha} \in E^\times \cap \mathbb{H}^n = \{z \in E^\times :\ \Phi(z) \in \mathbb{H}^n\}.$$

Then z represents a CM point in $X(\mathcal{O}_F)$ in the sense that $\mathbb{C}^n/\Lambda_z$ is a principally polarized abelian variety of type (E,Φ) with complex multiplication by $\mathcal{O}_E$. Conversely, every principally polarized abelian variety of type (E,Φ) with complex multiplication by $\mathcal{O}_E$ arises from a decomposition as in (4.1) for some $\mathfrak{a}$ in a unique fractional ideal class in $\mathrm{CL}(E)$. We denote the CM zero-cycle consisting of the set of CM points of type (E,Φ) by $\mathcal{CM}(E,\Phi,\mathcal{O}_F)$ and identify it with the set

$$\{z_{\mathfrak{a}} \in E^\times \cap \mathbb{H}^n : [\mathfrak{a}] \in \mathrm{CL}(E)\}$$

under the bijection just described. The reader should keep in mind that the latter set depends on Φ.

5. Periods of Eisenstein series

In this section we evaluate the non-holomorphic Hilbert modular Eisenstein series along a CM zero-cycle on the Hilbert modular variety $X(\mathcal{O}_F)$. Let F be a totally real number field of degree n over $\mathbb{Q}$ with narrow class number 1. Let E be a CM extension of F, and fix a CM type Φ for E. By the results of Section 3, given an ideal class $C \in \mathrm{CL}(E)$, there exists a fractional ideal $\mathfrak{a} \in C^{-1}$ such that

$$\mathfrak{a} = \mathcal{O}_F\alpha + \mathcal{O}_F\beta, \quad \alpha, \beta \in E^\times \tag{5.1}$$

where $z_\mathfrak{a} = \beta/\alpha \in E^\times \cap \mathbb{H}^n$ is a CM point of type (E, Φ).

By [M, Proposition 4.1], we have the identity

$$\zeta_E(s, C) = \left(\frac{2^n d_F}{\sqrt{d_E}}\right)^s \frac{1}{[\mathcal{O}_E^\times : \mathcal{O}_F^\times]} \zeta_F(2s) E(z_\mathfrak{a}, s),$$

where we have identified $z_\mathfrak{a}$ with its image $\Phi(z_\mathfrak{a}) \in \mathbb{H}^n$. Make the shift $s \mapsto (s+1)/2$ in this identity and sum over ideal classes $C \in \mathrm{CL}(E)$ to obtain

$$\sum_{[\mathfrak{a}] \in \mathrm{CL}(E)} E\left(z_\mathfrak{a}, \frac{s+1}{2}\right) = [\mathcal{O}_E^\times : \mathcal{O}_F^\times] \left(\frac{\sqrt{d_E}}{2^n d_F}\right)^{\frac{s+1}{2}} \frac{\zeta_E(\frac{s+1}{2})}{\zeta_F(s+1)}.$$

By class field theory, we have the factorization

$$\zeta_E(s) = \zeta_F(s) L(\chi_{E/F}, s), \tag{5.2}$$

where $L(\chi_{E/F}, s)$ is the L–function of the quadratic character $\chi_{E/F}$ associated to the extension E/F. Using the Taylor expansion (3.7), the factorization (5.2), and the Taylor expansion

$$\frac{\zeta_F\left(\frac{s+1}{2}\right)}{\zeta_F(s+1)} = \frac{1}{2^{n-1}} \left\{1 - \frac{1}{2n} \frac{\zeta_F^{(n)}(0)}{\zeta_F^{(n-1)}(0)}(s+1) + O((s+1)^2)\right\},$$

we obtain

$$\sum_{[\mathfrak{a}] \in \mathrm{CL}(E)} \left\{1 + \log\left(H(z_\mathfrak{a})\right)(s+1) + O(s+1)^2\right\} = \frac{[\mathcal{O}_E^\times : \mathcal{O}_F^\times] L(\chi_{E/F}, 0)}{2^n} \tag{5.3}$$

$$\times \left\{2 + \log\left(\frac{\sqrt{d_E}}{2^n d_F}\right)(s+1) - \frac{1}{n}\frac{\zeta_F^{(n)}(0)}{\zeta_F^{(n-1)}(0)}(s+1) + \frac{L'(\chi_{E/F}, 0)}{L(\chi_{E/F}, 0)}(s+1) + O((s+1)^2)\right\}.$$

Let $s = -1$ in (5.3) to recover the class number formula

$$L(\chi_{E/F}, 0) = \frac{2^{n-1} h_E}{[\mathcal{O}_E^\times : \mathcal{O}_F^\times]}.$$

Then differentiate (5.3) with respect to s and evaluate at $s = -1$ to get

$$\sum_{[\mathfrak{a}] \in \mathrm{CL}(E)} \log\left(H(z_\mathfrak{a})\right) = \frac{h_E}{2} \left\{\log\left(\frac{\sqrt{d_E}}{2^n d_F}\right) - \frac{1}{n}\frac{\zeta_F^{(n)}(0)}{\zeta_F^{(n-1)}(0)} + \frac{L'(\chi_{E/F}, 0)}{L(\chi_{E/F}, 0)}\right\}. \tag{5.4}$$

6. Evaluation of the logarithmic derivative

In this section we evaluate the logarithmic derivative of $L(\chi_{E/F}, s)$ at $s = 0$ in terms of values of the gamma function Γ at rational numbers. Let $\mathbb{Q} \subseteq F \subseteq E$ be abelian number fields. By the Kronecker-Weber theorem, there is a cyclotomic field $\mathbb{Q}(\zeta_N)$ such that $F \subseteq E \subseteq \mathbb{Q}(\zeta_N)$ where

$\zeta_N := e^{2\pi i/N}$ is a primitive N-th root of unity. Let $G_N := \mathrm{Gal}(\mathbb{Q}(\zeta_N)/\mathbb{Q})$, which we identify with the group $(\mathbb{Z}/N\mathbb{Z})^\times$ via the isomorphism

$$\begin{aligned} s_N : G_N &\longrightarrow (\mathbb{Z}/N\mathbb{Z})^\times \\ \sigma &\longmapsto [s_N(\sigma)]_N, \end{aligned}$$

where $\sigma(\zeta_N) = \zeta_N^{s_N(\sigma)}$ for some integer $s_N(\sigma)$ modulo N. Let H_F and H_E be the subgroups of G_N which fix F and E, resp. Since G_N is abelian, H_F and H_E are normal, and by Galois theory we have $\mathrm{Gal}(F/\mathbb{Q}) \cong G_N/H_F$ and $\mathrm{Gal}(E/\mathbb{Q}) \cong G_N/H_E$. We also note that $H_E \leq H_F \leq G_N$, since the Galois correspondence is inclusion reversing.

Let G be a finite abelian group and $\widehat{G}$ be its character group. Given a subgroup $H \leq G$, we have $\widehat{G/H} \cong H^\perp$ where

$$H^\perp := \{\chi \in \widehat{G} \mid \chi|_H \equiv 1\}.$$

Additionally, if $H' \leq H \leq G$ then $H^\perp \leq H'^\perp$.

Given an abelian field $K \subseteq \mathbb{Q}(\zeta_N)$, the group of characters associated to K is defined by

$$X_K := H_K^\perp = \{\chi \in \widehat{(\mathbb{Z}/N\mathbb{Z})^\times} \mid \chi|_{H_K} \equiv 1\}.$$

By our preceding observations, we have $\widehat{G_N/H_E} \cong X_E$ and $\widehat{G_N/H_F} \cong X_F$, and since $H_E \leq H_F \leq G_N$, we have $X_F \leq X_E$.

We now evaluate the logarithmic derivative of $L(\chi_{E/F}, s)$ at $s = 0$. The Dedekind zeta function $\zeta_K(s)$ of an abelian field $K \subset \mathbb{Q}(\zeta_N)$ factors as

$$\zeta_K(s) = \prod_{\chi \in X_K} L(\chi, s),$$

where $L(\chi, s)$ is understood to be the Dirichlet L–function associated to the primitive Dirichlet character of conductor c_χ which induces $\chi \in X_K$ (see [Coh, Theorem 10.5.25]). Therefore by (5.2), we have

$$\frac{L'(\chi_{E/F}, s)}{L(\chi_{E/F}, s)} = \frac{d}{ds}\left(\log \frac{\zeta_E(s)}{\zeta_F(s)}\right) = \sum_{\chi \in X_E \setminus X_F} \frac{L'(\chi, s)}{L(\chi, s)}, \tag{6.1}$$

where

$$X_E \setminus X_F = \{\chi \in \widehat{(\mathbb{Z}/N\mathbb{Z})^\times} \mid \chi|_{H_E} \equiv 1 \quad \text{and} \quad \chi|_{H_F \setminus H_E} \not\equiv 1\}$$

is the set of characters in $\widehat{(\mathbb{Z}/N\mathbb{Z})^\times}$ that are trivial on H_E but not trivial on H_F.

Now, we have

$$L(\chi, s) = c_\chi^{-s} \sum_{k=1}^{c_\chi} \chi(k) \zeta\left(s, \frac{k}{c_\chi}\right), \tag{6.2}$$

where

$$\zeta(s, w) := \sum_{n=0}^\infty \frac{1}{(n+w)^s}, \quad \mathrm{Re}(w) > 0, \quad \mathrm{Re}(s) > 1$$

is the Hurwitz zeta function. Differentiating (6.2) yields

$$L'(\chi, s) = -\log(c_\chi) L(\chi, s) + c_\chi^{-s} \sum_{k=1}^{c_\chi} \chi(k) \zeta'\left(s, \frac{k}{c_\chi}\right).$$

The Taylor expansion of the Hurwitz zeta function at $s = 0$ is given by

$$\zeta(s, x) = \zeta(0, x) + \zeta'(0, x)s + O(s^2), \quad x > 0$$

where $\zeta(0,x) = \frac{1}{2} - x$ and Lerch's identity [Le] gives

$$\zeta'(0,x) = \log\left(\frac{\Gamma(x)}{\sqrt{2\pi}}\right). \tag{6.3}$$

Using (6.3), we find that

$$L'(\chi,0) = -\log(c_\chi)L(\chi,0) + \sum_{k=1}^{c_\chi} \chi(k)\log\left(\frac{\Gamma\left(\frac{k}{c_\chi}\right)}{\sqrt{2\pi}}\right).$$

Recall that if χ is even, then $L(\chi,0)=0$, while if χ is odd, then $L(\chi,0)\neq 0$. If we assume that E is a CM extension of F, then all of the characters $\chi \in X_E \setminus X_F$ are odd (see Lemma 6.2). Hence using the orthogonality relations for group characters, we get

$$\frac{L'(\chi,0)}{L(\chi,0)} = -\log(c_\chi) + \frac{1}{L(\chi,0)}\sum_{k=1}^{c_\chi}\chi(k)\log\Gamma\left(\frac{k}{c_\chi}\right). \tag{6.4}$$

Finally, substituting (6.4) into (6.1) yields

$$\frac{L'(\chi_{E/F},0)}{L(\chi_{E/F},0)} = -\sum_{\chi\in X_E\setminus X_F}\log(c_\chi) + \sum_{\chi\in X_E\setminus X_F}\sum_{k=1}^{c_\chi}\frac{\chi(k)}{L(\chi,0)}\log\Gamma\left(\frac{k}{c_\chi}\right). \tag{6.5}$$

Remark 6.1. Since the primitive Dirichlet character χ of conductor c_χ which induces a Dirichlet character $\chi \in X_K$ is also a Dirichlet character modulo N, we have the following analog of (6.2),

$$L(\chi,s) = N^{-s}\sum_{k=1}^{N}\chi(k)\zeta\left(s,\frac{k}{N}\right). \tag{6.6}$$

Then by repeating the preceding calculation with (6.6) instead of (6.2), we get

$$\frac{L'(\chi_{E/F},0)}{L(\chi_{E/F},0)} = -\log(N)[F:\mathbb{Q}] + \sum_{\chi\in X_E\setminus X_F}\sum_{k=1}^{N}\frac{\chi(k)}{L(\chi,0)}\log\Gamma\left(\frac{k}{N}\right), \tag{6.7}$$

where we used $\#(X_E \setminus X_F) = [F:\mathbb{Q}]$. We will need (6.7) in the proof of Theorem 1.10.

It remains to prove the following

Lemma 6.2. *If E/F is a CM extension, then all of the characters $\chi \in X_E \setminus X_F$ are odd.*

Proof. Let E/F be a CM extension. Then the nontrivial automorphism $\sigma_c \in \mathrm{Gal}(E/F)$ is complex conjugation, which when viewed as an element of $G_N \cong (\mathbb{Z}/N\mathbb{Z})^\times$ corresponds to the residue class $[-1]_N \in (\mathbb{Z}/N\mathbb{Z})^\times$. Clearly, $[-1]_N \in H_F$ but $[-1]_N \notin H_E$, and by Galois theory we have $H_F = \langle H_E \cup \{[-1]_N\}\rangle$. Let $\chi \in X_E \setminus X_F$. Then χ is trivial on H_E but nontrivial on H_F, so we must have $\chi([-1]_N) = -1$, which implies that χ is odd. □

7. Taylor coefficients of Dedekind zeta functions

In this section we evaluate the logarithmic derivative of $\zeta_F^{(n-1)}(s)$ at $s=0$ and prove Theorem 1.1. The evaluation we obtain is analogous to (6.5), the difference being that $\log(\Gamma(x))$ is replaced by Deninger's R-function $R(x)$. Let F be a totally real field of degree n over $\mathbb{Q}$. Write the Laurent expansion of $\zeta_F(s)$ at $s=1$ as

$$\zeta_F(s) = \frac{A_{-1}}{s-1} + A_0 + O(s-1).$$

Lemma 7.1. *We have the Taylor expansion*

$$\zeta_F(s) = -\frac{\sqrt{d_F}A_{-1}}{2^n}s^{n-1} + \frac{\sqrt{d_F}}{2^n}(A_0 + A_{-1}\log(d_F) - nA_{-1}\{\gamma + \log(2\pi)\})s^n + O(s^{n+1}),$$

where γ is Euler's constant.

Proof. From the functional equation $\zeta_F^*(s) = \zeta_F^*(1-s)$, we have

$$\zeta_F(s) = d_F^{\frac{1}{2}-s}\left(\frac{\Gamma_{\mathbb{R}}(1-s)}{\Gamma_{\mathbb{R}}(s)}\right)^n \zeta_F(1-s),$$

where $\Gamma_{\mathbb{R}}(s) := \pi^{-s/2}\Gamma(s/2)$. Then the lemma follows by multiplying the Taylor expansions

$$d_F^{\frac{1}{2}-s} = \sqrt{d_F} - \sqrt{d_F}\log(d_F)s + O(s^2),$$

$$\begin{aligned}\left(\frac{\Gamma_{\mathbb{R}}(1-s)}{\Gamma_{\mathbb{R}}(s)}\right)^n &= \left(\frac{s}{2} + \frac{1}{2}\left(\gamma + \log(2\pi)\right)s^2 + O(s^3)\right)^n \\ &= \frac{s^n}{2^n} + \frac{n}{2^n}(\gamma + \log(2\pi))s^{n+1} + O(s^{n+2}),\end{aligned}$$

and

$$\zeta_F(1-s) = -\frac{A_{-1}}{s} + A_0 + O(s).$$

□

From Lemma 7.1, we have

$$\frac{\zeta_F^{(n-1)}(0)}{(n-1)!} = -\frac{\sqrt{d_F}A_{-1}}{2^n}$$

and

$$\frac{\zeta_F^{(n)}(0)}{n!} = \frac{\sqrt{d_F}}{2^n}\left(A_0 + A_{-1}\log(d_F) - nA_{-1}\{\gamma + \log(2\pi)\}\right),$$

which gives

$$\frac{\zeta_F^{(n)}(0)}{\zeta_F^{(n-1)}(0)} = -n\left(\frac{A_0}{A_{-1}} + \log(d_F) - n\gamma - n\log(2\pi)\right). \tag{7.1}$$

Assume now that F is abelian. Then we have the factorization

$$\zeta_F(s) = \zeta(s)\prod_{\substack{\chi\in X_F\\ \chi\neq 1}} L(\chi, s).$$

Substituting the Laurent expansions

$$\zeta(s) = \frac{1}{s-1} + \gamma + O(s-1)$$

and

$$L(\chi, s) = L(\chi, 1) + L'(\chi, 1)(s-1) + O((s-1)^2)$$

into this factorization yields

$$\zeta_F(s) = \left(\frac{1}{s-1} + \gamma + O(s-1)\right)\prod_{\substack{\chi\in X_F\\ \chi\neq 1}}\left(L(\chi, 1) + L'(\chi, 1)(s-1) + O((s-1)^2)\right).$$

Then expanding the right hand side and comparing coefficients yields

$$A_{-1} = \prod_{\substack{\chi \in X_F \\ \chi \neq 1}} L(\chi, 1)$$

and

$$A_0 = \gamma \prod_{\substack{\chi \in X_F \\ \chi \neq 1}} L(\chi, 1) + \left(\prod_{\substack{\chi \in X_F \\ \chi \neq 1}} L(\chi, 1) \right) \sum_{\substack{\chi \in X_F \\ \chi \neq 1}} \frac{L'(\chi, 1)}{L(\chi, 1)} = \gamma A_{-1} + A_{-1} \sum_{\substack{\chi \in X_F \\ \chi \neq 1}} \frac{L'(\chi, 1)}{L(\chi, 1)}.$$

It follows that

$$\frac{A_0}{A_{-1}} = \gamma + \sum_{\substack{\chi \in X_F \\ \chi \neq 1}} \frac{L'(\chi, 1)}{L(\chi, 1)}. \tag{7.2}$$

Each of the characters $\chi \in X_F$ is even, since $[-1]_N \in H_F$ and

$$X_F = \{\chi \in \widehat{(\mathbb{Z}/N\mathbb{Z})^\times} \mid \chi|_{H_F} \equiv 1\}.$$

Therefore, we must evaluate $L'(\chi, 1)$ for χ an even, primitive Dirichlet character. This problem was solved by Deninger [D] in the following way. Let χ be an even, primitive Dirichlet character of conductor c_χ. Then the functional equation for the Dirichlet L–function is

$$L(\chi, 1-s) = \frac{2c_\chi^{s-1}\Gamma(s)}{(2\pi)^s} \cos\left(\frac{\pi s}{2}\right) \tau(\chi) L(\overline{\chi}, s),$$

where

$$\tau(\chi) := \sum_{k=1}^{c_\chi} \chi(k) \zeta_{c_\chi}^k, \quad \zeta_{c_\chi} := e^{2\pi i / c_\chi}$$

is the Gauss sum of χ. A calculation with the functional equation yields

$$L'(\chi, 1) = \frac{2\tau(\chi)}{c_\chi} \left(\left(\gamma - \log\left(\frac{c_\chi}{2\pi}\right) \right) L'(\overline{\chi}, 0) - \frac{1}{2} L''(\overline{\chi}, 0) \right).$$

Because

$$L(\chi, s) = c_\chi^{-s} \sum_{k=1}^{c_\chi} \chi(k) \zeta\left(s, \frac{k}{c_\chi}\right),$$

to evaluate $L'(\overline{\chi}, 0)$ and $L''(\overline{\chi}, 0)$, it suffices to evaluate the coefficients in the Taylor expansion

$$\zeta(s, x) = \zeta(0, x) + \zeta'(0, x)s + \zeta''(0, x)s^2 + O(s^3), \quad x > 0.$$

Recall the logarithmic form of the Bohr-Mollerup theorem.

Theorem 7.2 (Bohr-Mollerup)**.** *Let $f : \mathbb{R}^+ \to \mathbb{R}$ be a function such that*

$$f(x+1) - f(x) = \log(x),$$

$f(1) = 0$, and $f(x)$ is convex on $\mathbb{R}^+$. Then $f(x) = \log(\Gamma(x))$.

Deninger [D, Theorem 2.2] proved the following result.

Theorem 7.3 (Deninger)**.** *The function*

$$f_\alpha(x) := (-1)^{\alpha+1} \left(\partial_s^\alpha \zeta(0, x) - \zeta^{(\alpha)}(0) \right), \quad x > 0, \quad \alpha = 0, 1, 2, \ldots$$

is the unique function such that

(1) $f_\alpha(x+1) - f_\alpha(x) = \log^\alpha(x)$
(2) $f_\alpha(1) = 0$
(3) $f_\alpha(x)$ *is convex on* $(\exp(\alpha-1), \infty)$.

Let $\alpha = 1$ in Theorem 7.3. Then $f_1(x)$ is convex on $(1,\infty)$ (hence convex on $\mathbb{R}^+$ by virtue of (1)), so by the Bohr-Mollerup theorem, $f_1(x) = \log(\Gamma(x))$, or equivalently

$$\zeta'(0,x) = \log\left(\frac{\Gamma(x)}{\sqrt{2\pi}}\right),$$

where we used $\zeta'(0) = -\frac{1}{2}\log(2\pi)$. This gives a conceptual proof of Lerch's identity (6.3) (a beautiful account of this approach to Lerch's identity is given by Weil [W, Chapter VII]). Moreover, using the limit

$$\Gamma(x) = \lim_{n\to\infty} \frac{n!n^x}{x(x+1)\cdots(x+n)}, \quad x > 0$$

one has

$$\log\left(\frac{\Gamma(x)}{\sqrt{2\pi}}\right) = \lim_{n\to\infty}\left(\zeta'(0) + x\log(n) - \log(x) - \sum_{k=1}^{n-1}(\log(x+k) - \log(k))\right).$$

Next, let $\alpha = 2$ in Theorem 7.3 and define $R(x) := -\zeta''(0,x)$. Then $R(x)$ is the unique function such that

(1′) $R(x+1) - R(x) = \log^2(x), \quad x > 0$
(2′) $R(1) = -\zeta''(0)$
(3′) $R(x)$ is convex on (e,∞).

Moreover, by [D, Lemma 2.1, eqn. (2.1.2)] one has

$$R(x) = \lim_{n\to\infty}\left(-\zeta''(0) + x\log^2(n) - \log^2(x) - \sum_{k=1}^{n-1}\left(\log^2(x+k) - \log^2(k)\right)\right). \tag{7.3}$$

These facts show that $R(x)$ is analogous to $\log(\Gamma(x)/\sqrt{2\pi})$ (see [D, Section 2] for more details concerning this analogy).

Remark 7.4. Alternatively, one could *define* $R(x)$ by the limit (7.3), then verify directly that $R(x)$ satisfies conditions (1′)–(3′). Then by uniqueness, one has the identity $R(x) = -\zeta''(0,x)$. This is analogous to the conceptual proof of Lerch's identity just described.

Using the preceding facts, Deninger [D, Section 3] established the formula

$$L'(\chi,1) = (\gamma + \log(2\pi))L(\chi,1) + \frac{\tau(\chi)}{c_\chi}\sum_{k=1}^{c_\chi}\overline{\chi}(k)R\left(\frac{k}{c_\chi}\right). \tag{7.4}$$

Substituting (7.4) into (7.2) yields

$$\frac{A_0}{A_{-1}} = \gamma + \sum_{\substack{\chi\in X_F\\ \chi\neq 1}}\left\{(\gamma + \log(2\pi)) + \frac{\tau(\chi)}{c_\chi}\sum_{k=1}^{c_\chi}\frac{\overline{\chi}(k)}{L(\chi,1)}R\left(\frac{k}{c_\chi}\right)\right\}. \tag{7.5}$$

Since $X_F \cong \widehat{G_N/H_F} \cong G_N/H_F \cong \mathrm{Gal}(F/\mathbb{Q})$, we have $\#X_F = [F : \mathbb{Q}] = n$. Then substituting (7.5) into (7.1) and simplifying yields the formula

$$\frac{\zeta_F^{(n)}(0)}{\zeta_F^{(n-1)}(0)} = -n\left(-\log(2\pi) + \log(d_F) + \sum_{\substack{\chi\in X_F\\ \chi\neq 1}} \frac{\tau(\chi)}{c_\chi}\sum_{k=1}^{c_\chi}\frac{\overline{\chi}(k)}{L(\chi,1)}R\left(\frac{k}{c_\chi}\right)\right). \tag{7.6}$$

Proof of Theorem 1.1. By combining equations (5.4), (6.5) and (7.6), we obtain Theorem 1.1 after a short calculation with the conductor-discriminant formula

$$d_L = \prod_{\chi\in X_L} c_\chi, \tag{7.7}$$

where d_L denotes the absolute value of the discriminant of a number field L. □

8. The group of characters of a multiquadratic extension

In this section we determine the group of characters associated to a multiquadratic extension. Let $d_1, \ldots, d_t$ be squarefree, pairwise relatively prime integers and define the multiquadratic extension $K = \mathbb{Q}(\sqrt{d_1}, \ldots, \sqrt{d_t})$. The absolute value of the discriminant of the quadratic subfield $\mathbb{Q}(\sqrt{d_i})$ is given by

$$D_i = \begin{cases} |d_i| & \text{if } d_i \equiv 1 \pmod 4 \\ 4|d_i| & \text{if } d_i \equiv 2, 3 \pmod 4). \end{cases}$$

One has $\mathbb{Q}(\sqrt{d_i}) \subseteq \mathbb{Q}(\zeta_{D_i})$, so by taking compositums we obtain

$$K = \mathbb{Q}(\sqrt{d_1}, \ldots, \sqrt{d_t}) \subseteq \mathbb{Q}(\zeta_{D_1}, \ldots, \zeta_{D_t}) \subseteq \mathbb{Q}(\zeta_{D_1\cdots D_t}) = \mathbb{Q}(\zeta_D)$$

where $D := D_1 \cdots D_t$.

Recall that the group of characters associated to K is given by

$$X_K = \{\chi \in \widehat{(\mathbb{Z}/D\mathbb{Z})^\times} \mid \chi|_{H_K} \equiv 1\},$$

where H_K is the subgroup of $G_D := \mathrm{Gal}(\mathbb{Q}(\zeta_D)/\mathbb{Q})$ which fixes K. Let $m = d_1^{e_1} \cdots d_t^{e_t}$ for $(0, \ldots, 0) \neq (e_1, \ldots, e_t) \in \{0, 1\}^t$, and define the quadratic subfield

$$\mathbb{Q}(\sqrt{m}) = \mathbb{Q}(\sqrt{d_1^{e_1} \cdots d_t^{e_t}}) \subset K.$$

Let χ_1 be the trivial character of $(\mathbb{Z}/D\mathbb{Z})^\times$, and χ'_m be the Dirichlet character of $(\mathbb{Z}/D\mathbb{Z})^\times$ induced by the Kronecker symbol χ_m associated to the quadratic field $\mathbb{Q}(\sqrt{m})$.

Proposition 8.1. *The group of characters associated to K is given by*

$$X_K = \{\chi_1\} \cup \left\{\chi'_m : \ m = d_1^{e_1} \cdots d_t^{e_t} \quad \text{for} \quad (0, \ldots, 0) \neq (e_1, \ldots, e_t) \in \{0, 1\}^t\right\}.$$

Proof. For notational convenience, let $G_m := \mathrm{Gal}(\mathbb{Q}(\sqrt{m})/\mathbb{Q})$, and let $H_m := H_{\mathbb{Q}(\sqrt{m})}$ be the subgroup of G_D which fixes $\mathbb{Q}(\sqrt{m})$. Define the integers

$$M = M_m := \begin{cases} |m| & \text{if } m \equiv 1 \pmod 4 \\ 4|m| & \text{if } m \equiv 2, 3 \pmod 4). \end{cases}$$

Clearly, the primitive Dirichlet characters $\chi_m : (\mathbb{Z}/M\mathbb{Z})^\times \to \{\pm 1\}$ induce $2^\ell - 1$ Dirichlet characters $\chi'_m : (\mathbb{Z}/D\mathbb{Z})^\times \to \{\pm 1\}$ by composing with the projections $\pi : (\mathbb{Z}/D\mathbb{Z})^\times \to (\mathbb{Z}/M\mathbb{Z})^\times$. Thus to show $\chi'_m \in X_K$, it suffices to show $\chi'_m|_{H_K} \equiv 1$. In fact, because $H_K \leq H_m$, it suffices to show $\chi'_m|_{H_m} \equiv 1$. We have the diagram

$$\begin{array}{ccc} H_K \leq H_m \leq G_D & \xrightarrow{s_D} & (\mathbb{Z}/D\mathbb{Z})^\times \\ \downarrow{\scriptstyle \mathbf{res}} & & \downarrow{\scriptstyle \pi} \\ G_M & \xrightarrow{s_M} & (\mathbb{Z}/M\mathbb{Z})^\times \\ \downarrow{\scriptstyle \mathbf{res}} & & \downarrow{\scriptstyle \chi_m} \\ G_m & \xrightarrow{\simeq} & \{\pm 1\} \end{array} \qquad \left((\mathbb{Z}/D\mathbb{Z})^\times \xrightarrow{\chi'_m} \{\pm 1\} \right)$$

where **res** is the restriction map, and s_D and s_M are the canonical isomorphisms. We will prove that

$$\chi'_m([s_D(\sigma)]_D) = \frac{\sigma(\sqrt{m})}{\sqrt{m}} \quad \text{for all} \quad \sigma \in G_D. \tag{8.1}$$

Then (8.1) implies that $\chi'_m|_{H_m} \equiv 1$, since

$$\frac{\sigma(\sqrt{m})}{\sqrt{m}} = 1 \quad \text{for all} \quad \sigma \in H_m.$$

That is, an automorphism $\sigma \in H_m$ restricts to the identity in G_m. Because the following diagram commutes (see [Ka, Proposition 5.14])

$$\begin{array}{ccc} G_M & \xrightarrow{s_M} & (\mathbb{Z}/M\mathbb{Z})^\times \\ \downarrow{\scriptstyle \mathbf{res}} & & \downarrow{\scriptstyle \chi_m} \\ G_m & \xrightarrow{\simeq} & \{\pm 1\} \end{array}$$

we have

$$\chi_m([s_M(\sigma)]_M) = \frac{\sigma(\sqrt{m})}{\sqrt{m}} \quad \text{for} \quad \sigma \in G_M.$$

Thus to prove (8.1), it suffices to show that

$$\chi'_m([s_D(\sigma)]_D) = \chi_m([s_M(\mathbf{res}(\sigma))]_M) \quad \text{for} \quad \sigma \in G_D.$$

Let $\sigma \in G_D$. Then since $\chi'_m = \chi_m \circ \pi$, we have $\chi'_m([s_D(\sigma)]_D) = \chi_m(\pi([s_D(\sigma)]_D)) = \chi_m([s_D(\sigma)]_M)$. Thus it suffices to show $[s_D(\sigma)]_M = [s_M(\mathbf{res}(\sigma))]_M$, or equivalently, $s_D(\sigma) \equiv s_M(\mathbf{res}(\sigma)) \pmod{M}$. Since $M|D$, there is an integer k such that $\zeta_M = \zeta_D^k$. Thus $\sigma(\zeta_M) = \sigma(\zeta_D^k) = \sigma(\zeta_D)^k = \zeta_D^{ks_D(\sigma)} = \zeta_M^{s_D(\sigma)}$. On the other hand, $\sigma(\zeta_M) = \mathbf{res}(\sigma)(\zeta_M) = \zeta_M^{s_M(\mathbf{res}(\sigma))}$, thus $s_D(\sigma) \equiv s_M(\mathbf{res}(\sigma)) \pmod{M}$. □

9. Proof of Theorem 1.4

In this section we prove Theorem 1.4. We first recall the setup in the theorem. Let $d_1, \ldots, d_{\ell+1}$ be squarefree, pairwise relatively prime integers with $d_i > 0$ for $i = 1, \ldots, \ell$ and $d_{\ell+1} < 0$, where $\ell = 1$ or 2. Assume that $F = \mathbb{Q}(\sqrt{d_1}, \ldots, \sqrt{d_\ell})$ has narrow class number 1, and let $E = F(\sqrt{d_{\ell+1}})$. Let χ_α (resp. χ_β) be the Kronecker symbol associated to the quadratic field $\mathbb{Q}(\sqrt{\alpha})$ (resp. $\mathbb{Q}(\sqrt{\beta})$), where $\alpha = d_1^{e_1} \cdots d_\ell^{e_\ell} d_{\ell+1}$ (resp. $\beta = d_1^{e_1} \cdots d_\ell^{e_\ell}$) for $(e_1, \ldots, e_\ell) \in \{0,1\}^\ell$. Now, the field F is totally real of degree $n = 2^\ell$ over $\mathbb{Q}$, and E is a CM extension of F. We have $F \subset E \subset \mathbb{Q}(\zeta_D)$ where $D = D_1 \cdots D_{\ell+1}$ (see Section 8 for the notation). Then by Proposition 8.1,

$$X_F = \{\chi_1\} \cup \left\{\chi'_\beta \in \widehat{(\mathbb{Z}/D\mathbb{Z})^\times} :\ \beta = d_1^{e_1}\cdots d_\ell^{e_\ell},\quad (0,\dots,0) \neq (e_1,\dots,e_\ell) \in \{0,1\}^\ell\right\} \quad \text{and}$$
$$X_E = \{\chi_1\} \cup \left\{\chi'_\alpha \in \widehat{(\mathbb{Z}/D\mathbb{Z})^\times} :\ \alpha = d_1^{e_1}\cdots d_{\ell+1}^{e_{\ell+1}},\quad (0,\dots,0) \neq (e_1,\dots,e_{\ell+1}) \in \{0,1\}^{\ell+1}\right\}.$$

It follows that

$$X_E \setminus X_F = \left\{\chi'_\alpha \in \widehat{(\mathbb{Z}/D\mathbb{Z})^\times} \mid \alpha = d_1^{e_1}\cdots d_\ell^{e_\ell} d_{\ell+1},\quad (e_1,\dots,e_\ell) \in \{0,1\}^\ell\right\}.$$

Using the class number formulas

$$L(\chi_\alpha, 0) = \frac{2h_\alpha}{w_\alpha} \quad \text{and} \quad L(\chi_\beta, 1) = \frac{2h_\beta \log \varepsilon_\beta}{\sqrt{c_\beta}},$$

along with the evaluation $\tau(\chi_\beta) = \sqrt{c_\beta}$, we deduce Theorem 1.4 from Theorem 1.1. □

10. Proof of Theorem 1.6

In this section we prove Theorem 1.6, which amounts to using the assumptions in Theorem 1.6 to give an explicit version of the formula appearing in Theorem 1.4 for a particular choice of CM point $z_{\mathcal{O}_E}$. We first recall the setup in the theorem. Let $p = 2$ or $p \equiv 1 \mod 4$ be a prime such that $F = \mathbb{Q}(\sqrt{p})$ has narrow class number 1. Let $d < 0$ be a squarefree integer relatively prime to p such that $E = \mathbb{Q}(\sqrt{p}, \sqrt{d})$ has class number 1. Let Δ_p, Δ_d and Δ_{pd} be the discriminants of $\mathbb{Q}(\sqrt{p}), \mathbb{Q}(\sqrt{d})$ and $\mathbb{Q}(\sqrt{pd})$, resp., and assume that Δ_p and Δ_d are relatively prime. The four embeddings of E are given by

$$\begin{aligned}
\mathrm{id}:&\quad \sqrt{p} \longmapsto \sqrt{p}, \quad \sqrt{d} \longmapsto \sqrt{d}\\
\sigma:&\quad \sqrt{p} \longmapsto -\sqrt{p}, \quad \sqrt{d} \longmapsto \sqrt{d}\\
\tau:&\quad \sqrt{p} \longmapsto \sqrt{p}, \quad \sqrt{d} \longmapsto -\sqrt{d}\\
\sigma\tau:&\quad \sqrt{p} \longmapsto -\sqrt{p}, \quad \sqrt{d} \longmapsto -\sqrt{d}.
\end{aligned}$$

These embeddings occur in the complex conjugate pairs $\{\mathrm{id}, \tau\}$ and $\{\sigma, \sigma\tau\}$. Fix the choice of CM type $\Phi = \{\mathrm{id}, \sigma\}$. We now determine a CM point of type (E, Φ) associated to the ideal class $[\mathcal{O}_E]$. Define θ_p and θ_d by

$$\theta_p := \begin{cases} \dfrac{1+\sqrt{p}}{2} & \text{if } p \equiv 1 \pmod 4 \\ \sqrt{2} & \text{if } p = 2 \end{cases} \quad \text{and} \quad \theta_d := \begin{cases} \dfrac{1+\sqrt{d}}{2} & \text{if } d \equiv 1 \pmod 4 \\ \sqrt{d} & \text{if } d \equiv 2,3 \pmod 4. \end{cases}$$

The integer rings $\mathcal{O}_F = \mathcal{O}_{\mathbb{Q}(\sqrt{p})}$ and $\mathcal{O}_{\mathbb{Q}(\sqrt{d})}$ have integral bases $\{1, \theta_p\}$ and $\{1, \theta_d\}$, resp. Since Δ_p and Δ_d are relatively prime, and $E = \mathbb{Q}(\sqrt{p}, \sqrt{d})$ is the compositum of $\mathbb{Q}(\sqrt{p})$ and $\mathbb{Q}(\sqrt{d})$, it follows that $\mathcal{O}_E$ has the integral basis $\{1, \theta_p, \theta_d, \theta_p\theta_d\}$ and $d_E = \Delta_p^2\Delta_d^2$ (see [L2, Chapter 3, Theorem 17]). Recall from Section 4 that to determine a CM point $z_{\mathcal{O}_E}$ of type (E, Φ) associated to the ideal class $[\mathcal{O}_E]$, we need a decomposition $\mathcal{O}_E = \mathcal{O}_F\alpha + \mathcal{O}_F\beta$ for some $\alpha, \beta \in \mathcal{O}_E$ with $\beta/\alpha \in E^\times \cap \mathbb{H}^2 = \{z \in E^\times :\ \Phi(z) \in \mathbb{H}^2\}$. We have

$$\mathcal{O}_E = \mathbb{Z} + \theta_p\mathbb{Z} + \theta_d\mathbb{Z} + \theta_p\theta_d\mathbb{Z} = (\mathbb{Z} + \theta_p\mathbb{Z}) + (\mathbb{Z} + \theta_p\mathbb{Z})\theta_d = \mathcal{O}_F + \mathcal{O}_F\theta_d.$$

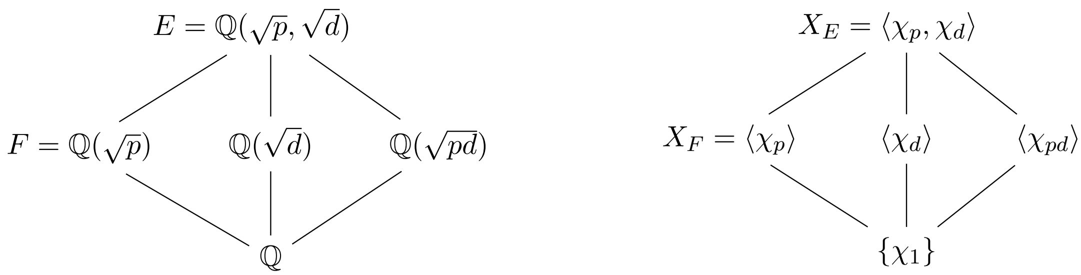
Letting $\alpha = 1$ and $\beta = \theta_d$, we get a CM point $z_{\mathcal{O}_E} = \beta/\alpha = \theta_d$, since $\Phi(\theta_d) = (\mathrm{id}(\theta_d), \sigma(\theta_d)) = (\theta_d, \theta_d) \in \mathbb{H}^2$. Then with our convention of identifying a CM point with its image under the CM type Φ, we have

$$z_{\mathcal{O}_E} = \Phi(\theta_d) = \begin{cases} (\sqrt{d}, \sqrt{d}), & d \equiv 2,3 \pmod 4 \\ \left(\frac{1+\sqrt{d}}{2}, \frac{1+\sqrt{d}}{2}\right), & d \equiv 1 \pmod 4). \end{cases}$$

To determine the constant $c_1(E,F,2)$, recall that $d_E = \Delta_p^2\Delta_d^2$, $d_F = \Delta_p$ and $h_E = 1$, thus

$$c_1(E,F,2) = \left(\frac{\Delta_p}{8\pi\sqrt{\Delta_p^2\Delta_d^2}}\right)^{\frac{1}{2}} = \frac{1}{2\sqrt{2\pi|\Delta_d|}}.$$

The groups of characters associated to the fields F and E are $X_F = \{\chi_1, \chi_p\}$ and $X_E = \{\chi_1, \chi_p, \chi_d, \chi_{pd}\}$, resp., so that $X_E \setminus X_F = \{\chi_d, \chi_{pd}\}$. The character $\chi_p = \left(\frac{\Delta_p}{\cdot}\right)$ has conductor Δ_p, the character $\chi_d = \left(\frac{\Delta_d}{\cdot}\right)$ has conductor $|\Delta_d|$, and the character $\chi_{pd} = \left(\frac{\Delta_{pd}}{\cdot}\right)$ has conductor $|\Delta_{pd}|$. The characters χ_p and χ_d generate X_E. The following diagrams show the correspondence between subfields and associated groups of characters:



Since $F = \mathbb{Q}(\sqrt{p})$ has narrow class number 1, we have $h_p = 1$. Then recalling that ε_p denotes the fundamental unit in F, the result follows by substituting the quantities determined in this section into the identity in Theorem 1.4. □

11. Faltings Heights of CM Abelian Varieties

In this section we review Colmez's conjecture and prove Theorem 1.7 and Proposition 1.8. We first recall the definition of the Faltings height following [Col, p. 667, (II.2.12.1)]. Let E be a CM extension of a totally real field F of degree n over $\mathbb{Q}$. Let A be an abelian variety with complex multiplication by E which is defined over $\overline{\mathbb{Q}}$. Let $K \subset \overline{\mathbb{Q}}$ be a number field over which A is defined and let $\omega_A \in H^0(A, \Omega_A^n)$ be a Néron differential. The Faltings height of A is defined by

$$h_{\mathrm{Fal}}(A) := -\frac{1}{[K:\mathbb{Q}]}\left(\sum_{\sigma\in \mathrm{Hom}(K,\overline{\mathbb{Q}})} \frac{1}{2}\log\left(\int_{A^\sigma(\mathbb{C})} |\omega_A^\sigma \wedge \overline{\omega_A^\sigma}|\right) - \sum_{p<\infty}\sum_{\sigma\in\mathrm{Hom}(K,\overline{\mathbb{Q}})} v_p(\omega_A^\sigma)\log(p)\right),$$

where $v_p(\omega_A^\sigma)$ is a certain rational number defined using the p-adic valuation on $\overline{\mathbb{Q}}_p$ (see [Col, p. 659]).

Let $\Phi(E)$ be the set of CM types for E, and given a type $\Phi \in \Phi(E)$, let A_Φ be a CM abelian variety of type $(\mathcal{O}_E, \Phi)$ defined over $\overline{\mathbb{Q}}$. Colmez [Col, equation (3)] conjectured the following identity for the average of the Faltings heights of the abelian varieties A_Φ,

$$\frac{1}{2^n}\sum_{\Phi\in\Phi(E)} h_{\mathrm{Fal}}(A_\Phi) = -\frac{1}{2}\left\{\frac{L'(\chi_{E/F},0)}{L(\chi_{E/F},0)} + \frac{1}{2}\log(\mathfrak{f}_{\chi_{E/F}}) + n\log(2\pi)\right\}, \tag{11.1}$$

where $\mathfrak{f}_{\chi_{E/F}}$ is the analytic Artin conductor of the quadratic character $\chi_{E/F}$ (here we have corrected a minor typographical error in the statement of [Col, equation (3)]). When $E/\mathbb{Q}$ is abelian, Colmez [Col, Théorème 5] proved the identity (11.1), up to addition by a possible rational multiple of $\log(2)$. Obus [O] recently completed Colmez's proof by eliminating this possible term.

We have the following result.

Proposition 11.1. *Let $F/\mathbb{Q}$ be a totally real field of degree n and E/F be a CM extension with $E/\mathbb{Q}$ abelian. Given a CM type $\Phi \in \Phi(E)$, let A_Φ be a CM abelian variety of type $(\mathcal{O}_E, \Phi)$ defined over $\overline{\mathbb{Q}}$. Then*

$$\prod_{\Phi\in\Phi(E)} \exp(h_{\mathrm{Fal}}(A_\Phi)) = \left(\frac{(2\pi)^n d_F \sqrt{\mathfrak{f}_{\chi_{E/F}}}}{d_E}\right)^{-2^{n-1}} \prod_{\chi\in X_E\setminus X_F} \prod_{k=1}^{c_\chi} \Gamma\left(\frac{k}{c_\chi}\right)^{\frac{-2^{n-1}\chi(k)}{L(\chi,0)}}.$$

Proof. From (11.1) and (6.5) we have

$$\sum_{\Phi\in\Phi(E)} h_{\mathrm{Fal}}(A_\Phi) = -2^{n-1}\left\{\log\left(\frac{(2\pi)^n d_F\sqrt{\mathfrak{f}_{\chi_{E/F}}}}{d_E}\right) + \sum_{\chi\in X_E\setminus X_F}\sum_{k=1}^{c_\chi}\frac{\chi(k)}{L(\chi,0)}\log\Gamma\left(\frac{k}{c_\chi}\right)\right\},$$

where we used the conductor-discriminant formula (7.7) to write

$$-\sum_{\chi\in X_E\setminus X_F}\log(c_\chi) = \log\left(\frac{d_F}{d_E}\right).$$

The result follows by exponentiating. □

Proof of Proposition 1.8. This follows by combining Proposition 11.1 with an argument similar to that in Section 9. □

Finally, we combine our results with (11.1) to prove Theorem 1.7, which evaluates the product of CM values $\prod_{[\mathfrak{a}]} H(z_{\mathfrak{a}})$ in terms of Faltings heights. This provides a geometric interpretation of the CM values by relating them to volumes of the complex manifolds $A_\Phi^\sigma(\mathbb{C})$.

Proof of Theorem 1.7. From (11.1) we have

$$\frac{L'(\chi_{E/F},0)}{L(\chi_{E/F},0)} = -\frac{1}{2^{n-1}}\sum_{\Phi\in\Phi(E)} h_{\mathrm{Fal}}(A_\Phi) - \log\left((2\pi)^n\sqrt{\mathfrak{f}_{\chi_{E/F}}}\right). \tag{11.2}$$

On the other hand, by (5.4) and (7.6) we have

$$\frac{L'(\chi_{E/F},0)}{L(\chi_{E/F},0)} = \frac{2}{h_E}\sum_{[\mathfrak{a}]\in\mathrm{CL}(E)}\log(H(z_{\mathfrak{a}})) - \log\left(\frac{\sqrt{d_E}}{2^{n+1}\pi d_F^2}\right) - \sum_{\substack{\chi\in X_F\\ \chi\neq 1}}\frac{\tau(\chi)}{c_\chi}\sum_{k=1}^{c_\chi}\frac{\overline{\chi}(k)}{L(\chi,1)}R\left(\frac{k}{c_\chi}\right). \tag{11.3}$$

The result follows by equating (11.2) and (11.3), then exponentiating. □

12. Faltings heights and the Lang-Rohrlich conjecture

In this section we review the Koblitz-Ogus criterion and prove Theorem 1.10. Recall that the Koblitz-Ogus criterion gives a sufficient condition for a product of gamma values at rational numbers to be algebraic, modulo an explicit rational power of π. Let N be a positive integer and consider the set $\frac{1}{N}\mathbb{Z}$. One can define an equivalence relation on $\frac{1}{N}\mathbb{Z}$ by

$$\frac{a}{N}\sim\frac{b}{N} \iff \frac{a}{N}-\frac{b}{N}\in\mathbb{Z}.$$

Write the corresponding quotient space as

$$\frac{1}{N}\mathbb{Z}\Big/\sim = \left\{\left[\frac{a}{N}\right] : 0\le a\le N-1\right\}.$$

Let A_N be the set of nonzero equivalence classes. The group $U_N = (\mathbb{Z}/N\mathbb{Z})^\times$ acts on the set A_N by

$$\bar{u} \cdot \left[\frac{a}{N}\right] := \left[\frac{ua}{N}\right], \quad \bar{u} \in U_N.$$

Given a function $f : A_N \to \mathbb{C}$, define the function $\langle f \rangle : U_N \to \mathbb{C}$ by

$$\langle f \rangle(\bar{u}) := \sum_{a=1}^{N-1} \frac{a}{N} f\left(\left[\frac{ua}{N}\right]\right).$$

We can now state the Koblitz-Ogus criterion (see the Appendix to [De]).

Theorem 12.1 (Koblitz-Ogus). *If $f : A_N \to \mathbb{Q}$ is a function such that $\langle f \rangle \equiv k \in \mathbb{Q}$ is constant, then*

$$\Gamma(f) := \pi^{-k} \prod_{a=1}^{N-1} \Gamma\left(\frac{a}{N}\right)^{f([\frac{a}{N}])} \in \overline{\mathbb{Q}}.$$

The converse of the Koblitz-Ogus theorem is the following conjecture of Lang and Rohrlich (see e.g. [L, Appendix to Section 2, p. 66] and the introduction to [ABP]).

Conjecture 12.2 (Lang-Rohrlich). *If $f : A_N \to \mathbb{Q}$ is a function such that $\langle f \rangle$ is not constant, then*

$$\Gamma(f) \notin \pi^k \overline{\mathbb{Q}}$$

for any $k \in \mathbb{Q}$.

We may now prove Theorem 1.10.

Proof of Theorem 1.10. We first explain how to obtain an alternative version of the identity in Proposition 1.8. Let χ_α be the Kronecker symbol associated to the quadratic subfield $\mathbb{Q}(\sqrt{\alpha})$, where $\alpha = d_1^{e_1} \cdots d_\ell^{e_\ell} d_{\ell+1}$ for $\mathbf{e} = (e_1, \ldots, e_\ell) \in \{0,1\}^\ell$. From Section 8, we know that $F \subset E \subset \mathbb{Q}(\zeta_D)$ where $D = D_1 \cdots D_{\ell+1}$, and that each character χ_α induces a Dirichlet character χ'_α modulo D. We use the identity (6.7) (with $N = D$) instead of (6.5) in the proof of Proposition 11.1, then argue as in the proof of Proposition 1.8 to obtain

$$\prod_{\Phi \in \Phi(E)} \exp(h_{\mathrm{Fal}}(A_\Phi)) = \left(\frac{(2\pi)^{2^\ell} \sqrt{\mathfrak{f}_{\chi_{E/F}}}}{D^{2^\ell}}\right)^{-2^{2^\ell - 1}} \prod_{\substack{\mathbf{e} \in \{0,1\}^\ell \\ \alpha = d_1^{e_1} \cdots d_\ell^{e_\ell} d_{\ell+1}}} \prod_{a=1}^{D} \Gamma\left(\frac{a}{D}\right)^{\frac{-2^{2^\ell - 2} \chi_\alpha(a) w_\alpha}{h_\alpha}}.$$

Now, we have

$$\prod_{\substack{\mathbf{e} \in \{0,1\}^\ell \\ \alpha = d_1^{e_1} \cdots d_\ell^{e_\ell} d_{\ell+1}}} \prod_{a=1}^{D} \Gamma\left(\frac{a}{D}\right)^{-\frac{2^{2^\ell - 2} \chi_\alpha(a) w_\alpha}{h_\alpha}} = \prod_{a=1}^{D-1} \Gamma\left(\frac{a}{D}\right)^{-2^{2^\ell - 2} \sum\limits_{\substack{\mathbf{e} \in \{0,1\}^\ell \\ \alpha = d_1^{e_1} \cdots d_\ell^{e_\ell} d_{\ell+1}}} \frac{\chi_\alpha(a) w_\alpha}{h_\alpha}}. \tag{12.1}$$

Define a function $f : A_D \to \mathbb{Q}$ by

$$f\left(\left[\frac{a}{D}\right]\right) := -2^{2^\ell - 2} \sum_{\substack{\mathbf{e} \in \{0,1\}^\ell \\ \alpha = d_1^{e_1} \cdots d_\ell^{e_\ell} d_{\ell+1}}} \frac{\chi_\alpha(a) w_\alpha}{h_\alpha}.$$

Since χ_α is periodic modulo D, this function is well-defined. Write the product (12.1) as

$$\Gamma(f) := \prod_{a=1}^{D-1} \Gamma\left(\frac{a}{D}\right)^{f([\frac{a}{D}])}.$$

Then assuming the Lang-Rohrlich conjecture, to complete the proof it suffices to show that $\langle f \rangle : U_D \to \mathbb{Q}$ is not constant. We will do this by showing that $\langle f \rangle(\bar{1}) > 0$ and $\langle f \rangle(\overline{-1}) < 0$.

We calculate

$$\begin{aligned}\langle f \rangle(\bar{u}) = \frac{1}{D}\sum_{a=1}^{D-1} af\left(\left[\frac{ua}{D}\right]\right) &= \frac{-2^{2^\ell-2}}{D}\sum_{a=1}^{D-1} \sum_{\substack{\mathbf{e}\in\{0,1\}^\ell \\ \alpha=d_1^{e_1}\cdots d_\ell^{e_\ell}d_{\ell+1}}} \frac{a\chi_\alpha(u)\chi_\alpha(a)w_\alpha}{h_\alpha} \\ &= \frac{-2^{2^\ell-2}}{D} \sum_{\substack{\mathbf{e}\in\{0,1\}^\ell \\ \alpha=d_1^{e_1}\cdots d_\ell^{e_\ell}d_{\ell+1}}} \frac{\chi_\alpha(u)w_\alpha}{h_\alpha} S_{D,\alpha},\end{aligned}$$

where

$$S_{D,\alpha} := \sum_{a=1}^{D-1} a\chi_\alpha(a).$$

We now show that $S_{D,\alpha} < 0$ for each α. The absolute value of the discriminant of $\mathbb{Q}(\sqrt{\alpha})$ is given by

$$M_\alpha = \begin{cases} |\alpha| & \text{if } \alpha \equiv 1 \pmod 4 \\ 4|\alpha| & \text{if } \alpha \equiv 2,3 \pmod 4. \end{cases}$$

By the Dirichlet class number formula, we have

$$h_\alpha = -\frac{w_\alpha}{2M_\alpha}\sum_{a=1}^{M_\alpha} a\chi_\alpha(a).$$

Since $M_\alpha | D$, we may write $D = b_\alpha M_\alpha$ for some integer $b_\alpha \geq 1$. Then using the decomposition

$$[1, D] = \bigcup_{j=0}^{b_\alpha-1} [jM_\alpha + 1, (j+1)M_\alpha],$$

we get

$$S_{D,\alpha} = \sum_{a=1}^{D-1} a\chi_\alpha(a) = \sum_{a=1}^{D} a\chi_\alpha(a) = \sum_{j=0}^{b_\alpha-1}\sum_{a=jM_\alpha+1}^{(j+1)M_\alpha} a\chi_\alpha(a),$$

where we used that χ_α is a Dirichlet character modulo M_α. The orthogonality relations for group characters yield

$$\sum_{a=jM_\alpha+1}^{(j+1)M_\alpha} a\chi_\alpha(a) = \sum_{a=1}^{M_\alpha}(a+jM_\alpha)\chi_\alpha(a+jM_\alpha) = \sum_{a=1}^{M_\alpha}(a+jM_\alpha)\chi_\alpha(a) = \sum_{a=1}^{M_\alpha} a\chi_\alpha(a).$$

Therefore, we get

$$S_{D,\alpha} = b_\alpha\sum_{a=1}^{M_\alpha} a\chi_\alpha(a) = -\frac{2h_\alpha(b_\alpha M_\alpha)}{w_\alpha} = -\frac{2h_\alpha D}{w_\alpha} < 0.$$

Finally, we have

$$\langle f\rangle(\bar{1}) = -\frac{2^{2^{\ell}-2}}{D} \sum_{\substack{\mathbf{e}\in\{0,1\}^{\ell} \\ \alpha=d_1^{e_1}\cdots d_\ell^{e_\ell} d_{\ell+1}}} \frac{w_\alpha}{h_\alpha} S_{D,\alpha} > 0,$$

and since the characters χ_α are odd, we have $\chi_\alpha(-1) = -1$, so that

$$\langle f\rangle(\overline{-1}) = \frac{2^{2^{\ell}-2}}{D} \sum_{\substack{\mathbf{e}\in\{0,1\}^{\ell} \\ \alpha=d_1^{e_1}\cdots d_\ell^{e_\ell} d_{\ell+1}}} \frac{w_\alpha}{h_\alpha} S_{D,\alpha} < 0.$$

We conclude that $\langle f\rangle : U_D \to \mathbb{Q}$ is not constant. □

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Department of Mathematics, Mailstop 3368, Texas A&M University, College Station, TX 77843-3368

E-mail address: `adrianbs11@math.tamu.edu`

E-mail address: `masri@math.tamu.edu`