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Faltings heights of CM elliptic curves and special Gamma values

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Abstract

In this paper, we evaluate the Faltings height of an elliptic curve with complex multiplication by an order in an imaginary quadratic field in terms of Euler's Gamma function at rational arguments.

1 Background

In the Seminar Bourbaki article [5], Deligne used the Chowla–Selberg formula [2] to evaluate the stable Faltings height of an elliptic curve with complex multiplication by the ring of integers $\mathcal{O}_K$ of an imaginary quadratic field K in terms of Euler's Gamma function $\Gamma(s)$ at rational arguments. He then used this result to calculate the minimum value attained by the stable Faltings height. In this paper, we will establish a similar formula for both the unstable and stable Faltings height of an elliptic curve with complex multiplication by any order in K (not necessarily maximal). We illustrate these results by explicitly evaluating the Faltings height of an elliptic curve over $\mathbb{Q}$ with complex multiplication by a non-maximal order (see Sect. 2).

We begin by recalling the definition of the (unstable) Faltings height of an elliptic curve, following ([12], Chapter IV, Sect. 6). Let L be a number field with ring of integers $\mathcal{O}_L$. Let E/L be an elliptic curve over L, and let $\mathcal{E}/\mathcal{O}_L$ be a Néron model for E/L. Let $\Omega_{\mathcal{E}/\mathcal{O}_L}$ be the sheaf of Néron differentials, and let $s^*\Omega_{\mathcal{E}/\mathcal{O}_L}$ be the pullback by the zero section $s : \mathrm{Spec}(\mathcal{O}_L) \to \mathcal{E}$. Choose a differential $\omega \in H^0(E/L, \Omega_{E/L})$. Then the *Faltings height* of E/L is defined by

$$h_{\mathrm{Fal}}(E/L) := \frac{\log(\#(s^*\Omega_{\mathcal{E}/\mathcal{O}_L}/\mathcal{O}_L\omega))}{[L:\mathbb{Q}]} - \frac{1}{2[L:\mathbb{Q}]} \sum_{\sigma: L\hookrightarrow\mathbb{C}} \log\left(\frac{i}{2}\int_{E^\sigma(\mathbb{C})} \omega^\sigma \wedge \overline{\omega^\sigma}\right).$$

The definition of the Faltings height given here is normalized as in Silverman [16] (who in turn uses the same normalization as Faltings [6, p. 14]).

To state our main results, we fix the following notation. Let K be an imaginary quadratic field of discriminant D with ideal class group $\mathrm{Cl}(D)$, unit group $\mathcal{O}_K^\times$, and Kronecker symbol χ_D. Let $h(D) = \#\mathrm{Cl}(D)$ be the class number and $w_D = \#\mathcal{O}_K^\times$ be the number of units. For an elliptic curve E/L, let $\Delta_{E/L}$ be the minimal discriminant ideal and $j(E)$ be the j-invariant.

Theorem 1.1 *Let E/L be an elliptic curve with complex multiplication by an order $\mathcal{O}_f \subset K$ of conductor $f \in \mathbb{Z}^+$ and discriminant $\Delta_f = f^2 D$. Assume that the coefficients of the Weierstrass equation for E/L are contained in $\mathbb{Q}(j(E))$. Then the Faltings height of E/L is given by*

$$\exp[h_{\mathrm{Fal}}(E/L)] = N_{L/\mathbb{Q}}(\Delta_{E/L})^{1/12[L:\mathbb{Q}]} \left(\frac{\sqrt{|\Delta_f|}}{\pi}\right)^{1/2} \times \prod_{k=1}^{|D|} \Gamma\left(\frac{k}{|D|}\right)^{-\chi_D(k) w_D/4h(D)} \prod_{p|f} p^{e(p)/2},$$

where

$$e(p) := \frac{(1 - p^{\mathrm{ord}_p(f)})(1 - \chi_D(p))}{p^{\mathrm{ord}_p(f)-1}(1-p)(\chi_D(p) - p)}.$$

Remark 1.2 Our assumption that the coefficients of the Weierstrass equation for E/L be contained in $\mathbb{Q}(j(E))$ is used crucially in the proof of Proposition 6.1, which is an important component in the proof of Theorem 1.1. This hypothesis can be removed if we instead work with the *stable* Faltings height, which we do in Theorem 1.3.

If L' is a finite extension of L, then it is not necessarily true that $h_{\mathrm{Fal}}(E/L) = h_{\mathrm{Fal}}(E/L')$. However, if an elliptic curve over a number field has everywhere semistable reduction, then the Faltings height is invariant under finite field extensions. This leads one to define the *stable Faltings height* of E/L by

$$h_{\mathrm{Fal}}^{\mathrm{stab}}(E/L) := h_{\mathrm{Fal}}(E/L'),$$

where L' is any finite extension of L such that E/L' has everywhere semistable reduction.

Theorem 1.3 *Let E/L be an elliptic curve with complex multiplication by an order $\mathcal{O}_f \subset K$ of conductor $f \in \mathbb{Z}^+$ and discriminant $\Delta_f = f^2 D$. Then the stable Faltings height of E/L is given by*

$$\exp[h_{\mathrm{Fal}}^{\mathrm{stab}}(E/L)] = \left(\frac{\sqrt{|\Delta_f|}}{\pi}\right)^{1/2} \prod_{k=1}^{|D|} \Gamma\left(\frac{k}{|D|}\right)^{-\chi_D(k) w_D/4h(D)} \prod_{p|f} p^{e(p)/2}.$$

Remark 1.4 We now briefly explain how Theorem 1.3 can be used to recover Deligne's evaluation of the stable Faltings height in the case that E/L has complex multiplication by the maximal order $\mathcal{O}_K$ in K. Since $\mathcal{O}_K$ has conductor $f = 1$, by Theorem 1.3 we have

$$\exp[h_{\mathrm{Fal}}^{\mathrm{stab}}(E/L)] = \left(\frac{\sqrt{|D|}}{\pi}\right)^{1/2} \prod_{k=1}^{|D|} \Gamma\left(\frac{k}{|D|}\right)^{-\chi_D(k) w_D/4h(D)}. \tag{1.1}$$

Now, Deligne [5, p. 27] defined a different normalization of the stable Faltings height which he called the *geometric height* of E and denoted by $h_{\mathrm{geom}}(E)$. It can be shown that

$$h_{\mathrm{geom}}(E) = h_{\mathrm{Fal}}^{\mathrm{stab}}(E/L) + \frac{1}{2}\log \pi. \tag{1.2}$$

Deligne [5, p. 29] then observed that the Chowla–Selberg formula [2] can be used to establish the identity

$$\exp[h_{\mathrm{geom}}(E)]^{-2} = \frac{1}{\sqrt{|D|}} \prod_{k=1}^{|D|} \Gamma\left(\frac{k}{|D|}\right)^{\chi_D(k)w_D/2h(D)}. \tag{1.3}$$

By substituting the evaluation of $h_{\mathrm{Fal}}^{\mathrm{stab}}(E/L)$ from (1.1) into (1.2), we recover Deligne's result (1.3).

An important component in the proof of Theorem 1.1 is a Chowla–Selberg formula for any order in K. An arithmetic-geometric proof of such a formula was given by Nakkajima and Taguchi [13] by employing a theorem of Faltings which relates the Faltings heights of two isogenous abelian varieties. Kaneko briefly outlined an analytic approach to the same formula in the research announcement [7]. Here we give a detailed analytic proof of a Chowla–Selberg formula for orders in K. This proof is based on a renormalized Kronecker limit formula for the non-holomorphic Eisenstein series on $SL_2(\mathbb{Z})$, a period formula which relates the zeta function of an order in K to values of the Eisenstein series at CM points corresponding to classes in the ideal class group of the order, and a factorization of the zeta function of an order given by Zagier [19], and in an equivalent but different form by Kaneko [7].

2 Examples

In this section, we use Theorems 1.1 and 1.3, and `SageMath` [14] to evaluate the unstable and stable Faltings height of an elliptic curve over $\mathbb{Q}$ with complex multiplication by a non-maximal order.

Example 2.1 Let $K = \mathbb{Q}(\sqrt{-7})$ be the imaginary quadratic field of discriminant $D = -7$. Let $\mathcal{O}_K = \mathbb{Z}\left[\frac{1+\sqrt{-7}}{2}\right]$ be the ring of integers, and let

$$\mathcal{O}_2 = \mathbb{Z} + 2\mathcal{O}_K = \mathbb{Z}[1+\sqrt{-7}]$$

be the order of conductor $f = 2$ in K. Let $A = 255$ and consider the elliptic curve (see [8, Eq. (2.2)])

$$E_A/\mathbb{Q} : y^2 = x^3 - 3A(A^3 - 1728)x - 2(A^3 - 1728)^2.$$

The elliptic curve $E_A/\mathbb{Q}$ has complex multiplication by the non-maximal order $\mathcal{O}_2$ and the minimal discriminant ideal is

$$\Delta_{E_A/\mathbb{Q}} = (3^6 \cdot 7^3 \cdot 19^6)\mathbb{Z}.$$

Moreover, the j-invariant of $E_A/\mathbb{Q}$ is $j = A^3 = 255^3$.

We now use Theorems 1.1 and 1.3 to evaluate the unstable and stable Faltings height of $E_A/\mathbb{Q}$.

Since the discriminant of K is $D = -7$ and the conductor of the order $\mathcal{O}_2$ is $f = 2$, we have $\Delta_2 = -28$. Also, $w_{-7} = 2$ and $h(-7) = 1$. The Kronecker symbol values are

$\chi_{-7}(k) = 1$ for $k = 1, 2, 4$ and $\chi_{-7}(k) = -1$ for $k = 3, 5, 6$. The only prime $p|f$ is $p = 2$, and we have

$$e(2) = \frac{(1-2)(1-\chi_{-7}(2))}{2^{1-1}(1-2)(\chi_{-7}(2)-2)} = 0.$$

Then since the Weierstrass equation defining $E_A/\mathbb{Q}$ has coefficients in $\mathbb{Q}(j) = \mathbb{Q}$, by Theorem 1.1 the Faltings height of $E_A/\mathbb{Q}$ is given by

$$\exp[h_{\text{Fal}}(E_A/\mathbb{Q})] = N_{\mathbb{Q}/\mathbb{Q}}(\Delta_{E_A/\mathbb{Q}})^{1/12} \left(\frac{2\sqrt{7}}{\pi}\right)^{1/2} \prod_{k=1}^{7} \Gamma\left(\frac{k}{7}\right)^{-\chi_{-7}(k)/2}.$$

After further simplification, we get

$$\exp[h_{\text{Fal}}(E_A/\mathbb{Q})] = 3^{1/2}7^{1/4}19^{1/2} \left(\frac{2\sqrt{7}}{\pi}\right)^{1/2} \left(\frac{\Gamma(3/7)\Gamma(5/7)\Gamma(6/7)}{\Gamma(1/7)\Gamma(2/7)\Gamma(4/7)}\right)^{1/2}.$$

Similarly, using the preceding computations, by Theorem 1.3 the stable Faltings height of $E_A/\mathbb{Q}$ is given by

$$\exp[h_{\text{Fal}}^{\text{stab}}(E_A/\mathbb{Q})] = \left(\frac{2\sqrt{7}}{\pi}\right)^{1/2} \left(\frac{\Gamma(3/7)\Gamma(5/7)\Gamma(6/7)}{\Gamma(1/7)\Gamma(2/7)\Gamma(4/7)}\right)^{1/2}.$$

Numerically, the values of the Faltings heights computed above are $h_{\text{Fal}}(E_A/\mathbb{Q}) \approx 1.56896083514163$ and $h_{\text{Fal}}^{\text{stab}}(E_A/\mathbb{Q}) \approx -0.939042336039478$.

Now, let $L = \mathbb{Q}(\sqrt{7})$ and

$$u = 1197 - 456\sqrt{7} \in L^\times \setminus L^{\times 2}.$$

Let E_A/L denote the base change of $E_A/\mathbb{Q}$ to L (given by the same Weierstrass equation). Then the quartic twist of $E_A/\mathbb{Q}$ by u is the elliptic curve[1]

$$E_A^u/L(\sqrt{u}) : y^2 = x^3 - 3A(A^3 - 1728)u^2x - 2(A^3 - 1728)^2u^3$$

over the quartic number field $L(\sqrt{u}) = \mathbb{Q}(\sqrt{1197 - 456\sqrt{7}})$. Note that the quartic twist $E_A^u/L(\sqrt{u})$ of $E_A/\mathbb{Q}$ is precisely the quadratic twist by $u = 1197 - 456\sqrt{7}$ of the base change E_A/L.

The minimal discriminant ideal of $E_A^u/L(\sqrt{u})$ is

$$\Delta_{E_A^u/L(\sqrt{u})} = \mathcal{O}_{L(\sqrt{u})},$$

hence the quartic twist $E_A^u/L(\sqrt{u})$ has everywhere good reduction. It follows that the stable Faltings height of $E_A/\mathbb{Q}$ is given by

$$h_{\text{Fal}}^{\text{stab}}(E_A/\mathbb{Q}) = h_{\text{Fal}}(E_A^u/L(\sqrt{u})).$$

However, since the coefficients of the Weierstrass equation of $E_A^u/L(\sqrt{u})$ are *not* contained in $\mathbb{Q}(j) = \mathbb{Q}$, we cannot apply Theorem 1.1 directly to evaluate the Faltings height of $E_A^u/L(\sqrt{u})$. This demonstrates the usefulness of Theorem 1.3 in evaluating the stable Faltings height of a CM elliptic curve.

[1]The elliptic curve E_A and its quartic twist E_A^u are taken from the third entry in [8, Table 3, p. 556].

3 The Kronecker limit formula

In this section, we briefly recall a renormalized version of the Kronecker (first) limit formula. Let $\mathbb{H}$ denote the complex upper half-plane and define the group

$$\Gamma_\infty := \left\{ \pm \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} : n \in \mathbb{Z} \right\} < SL_2(\mathbb{Z}).$$

Then the non-holomorphic Eisenstein series on $SL_2(\mathbb{Z})$ is defined by

$$E(z,s) := \sum_{M \in \Gamma_\infty \backslash SL_2(\mathbb{Z})} \operatorname{Im}(Mz)^s, \quad z = x+iy \in \mathbb{H}, \quad \operatorname{Re}(s) > 1.$$

The Eisenstein series has the well-known Fourier expansion (see e.g. [9, Theorem 9.9 (2), p. 32])

$$E(z,s) = y^s + \sqrt{\pi}\frac{\Gamma(s-\frac{1}{2})}{\Gamma(s)}\frac{\zeta(2s-1)}{\zeta(2s)}y^{1-s} + \frac{4\pi^s}{\Gamma(s)\zeta(2s)}\sqrt{y}\sum_{n=1}^{\infty}\sigma_{1-2s}(n)n^{s-\frac{1}{2}}K_{s-\frac{1}{2}}(2\pi ny)\cos(2\pi nx),$$

where $\Gamma(s)$ is Euler's Gamma function, $\zeta(s)$ is the Riemann zeta function, $\sigma_k(n) := \sum_{\ell|n}\ell^k$ is the k-divisor function, and K_ν is the K-Bessel function of order ν. The Fourier expansion shows that $E(z,s)$ extends to a meromorphic function on $\mathbb{C}$ with a simple pole at $s=1$.

Let $s \mapsto (s+1)/2$ in the Fourier expansion of $E(z,s)$ and calculate the Taylor expansion of the shifted Eisenstein series $E(z,(s+1)/2)$ at $s=-1$ to get

$$E\left(z,(s+1)/2\right) = 1 + \left(\log(\sqrt{y}) - \frac{\pi}{6}y - 2\sum_{n=1}^{\infty}\frac{\sigma_1(n)}{n}e^{-2\pi ny}\cos(2\pi nx)\right)(s+1) + O((s+1)^2).$$

Now, recall that the *Dedekind eta function* is the weight $1/2$ modular form for $SL_2(\mathbb{Z})$ defined by the infinite product

$$\eta(z) := q^{1/24}\prod_{n=1}^{\infty}(1-q^n), \quad q := e^{2\pi iz}, \quad z \in \mathbb{H}.$$

Then using the identity (see e.g. [10, p. 274])

$$\log(\sqrt{y}|\eta(z)|^2) = \log(\sqrt{y}) - \frac{\pi}{6}y - 2\sum_{n=1}^{\infty}\frac{\sigma_1(n)}{n}e^{-2\pi ny}\cos(2\pi nx),$$

one gets the following renormalized version of the Kronecker limit formula,

$$E\left(z,(s+1)/2\right) = 1 + \log(F(z))(s+1) + O((s+1)^2), \tag{3.1}$$

where $F(z)$ is the $SL_2(\mathbb{Z})$-invariant function defined by

$$F(z) := \sqrt{\operatorname{Im}(z)}|\eta(z)|^2. \tag{3.2}$$

4 Zeta functions of orders and CM values of Eisenstein series

In this section, we relate the zeta function of an order in an imaginary quadratic field to values of the Eisenstein series $E(z, s)$ at CM points corresponding to classes in the ideal class group of the order.

We begin by recalling some facts regarding orders in imaginary quadratic fields (see e.g. Cox [3, Sect. 7]). Let K be an imaginary quadratic field of discriminant D. Given $f \in \mathbb{Z}^+$, let $\mathcal{O}_f$ be the (unique) order of conductor f in K. A fractional $\mathcal{O}_f$-ideal $\mathfrak{a}$ is a subset of K which is a non-zero finitely generated $\mathcal{O}_f$-module. A fractional $\mathcal{O}_f$-ideal $\mathfrak{a}$ is *proper* if

$$\mathcal{O}_f = \{\beta \in K : \beta\mathfrak{a} \subset \mathfrak{a}\}.$$

It is known that a fractional $\mathcal{O}_f$-ideal is invertible if and only if it is proper (see [3, Proposition 7.4]). Accordingly, let $I(\mathcal{O}_f)$ be the group of proper fractional $\mathcal{O}_f$-ideals, and let $P(\mathcal{O}_f)$ be the subgroup of $I(\mathcal{O}_f)$ consisting of principal fractional $\mathcal{O}_f$-ideals. The *ideal class group* of $\mathcal{O}_f$ is defined as the quotient group

$$\mathrm{Cl}(\mathcal{O}_f) := I(\mathcal{O}_f)/P(\mathcal{O}_f).$$

Let $h(\mathcal{O}_f) = \#\,\mathrm{Cl}(\mathcal{O}_f)$ be the class number of $\mathcal{O}_f$.

The *Dedekind zeta function* of $\mathcal{O}_f$ is defined by

$$\zeta_{\mathcal{O}_f}(s) := \sum_{\substack{\mathfrak{a}\in I(\mathcal{O}_f)\\ \mathfrak{a}\subset\mathcal{O}_f}} \frac{1}{N(\mathfrak{a})^s}, \quad \mathrm{Re}(s) > 1.$$

Similarly, given an ideal class $A \in \mathrm{Cl}(\mathcal{O}_f)$, we define the *ideal class zeta function* by

$$\zeta_A(s) := \sum_{\substack{I\in A\\ I\subset\mathcal{O}_f}} \frac{1}{N(I)^s}, \quad \mathrm{Re}(s) > 1.$$

Then we have the decomposition

$$\zeta_{\mathcal{O}_f}(s) = \sum_{A\in\mathrm{Cl}(\mathcal{O}_f)} \zeta_A(s).$$

Now, the discriminant of $\mathcal{O}_f$ is given by $\Delta_f = f^2 D$. By [3, Theorem 7.7], we may choose a proper integral ideal $\mathfrak{a} \in A$ with

$$\mathfrak{a} = \mathbb{Z}a + \mathbb{Z}\left(\frac{-b+\sqrt{\Delta_f}}{2}\right)$$

where $[a, b, c](X, Y) = aX^2 + bXY + cY^2$ is a quadratic form of discriminant $b^2 - 4ac = \Delta_f$ with $(a, b, c) = 1$ and $a = N(\mathfrak{a}) > 0$.

For $\alpha \in K$, let α' denote the image of α under the nontrivial automorphism of K. Then

$$\mathfrak{a}' = \mathbb{Z}a + \mathbb{Z}\left(\frac{b+\sqrt{\Delta_f}}{2}\right).$$

Moreover, by [3, Eq. (7.6)] we have $\mathfrak{a}^{-1} = \frac{1}{a}\mathfrak{a}'$, and thus

$$\mathfrak{a}^{-1} = \mathbb{Z} + \mathbb{Z}\left(\frac{b+\sqrt{\Delta_f}}{2a}\right) = \mathbb{Z} + \mathbb{Z}z_{\mathfrak{a}^{-1}} \tag{4.1}$$

where

$$z_{\mathfrak{a}^{-1}} := \frac{b+\sqrt{\Delta_f}}{2a} \in \mathbb{H}$$

is the root in the complex upper half-plane of the dehomogenized form $[a, -b, c](X, 1) = aX^2 - bX + c$.

Let $\mathcal{O}_f^\times$ be the group of units in $\mathcal{O}_f$, and let $w_f = \#\mathcal{O}_f^\times$.

Proposition 4.1 *With notation as above, we have*

$$\zeta_{[\mathfrak{a}]}(s) = \frac{2}{w_f}\left(\frac{\sqrt{|\Delta_f|}}{2}\right)^{-s}\zeta(2s)E(z_{\mathfrak{a}^{-1}}, s).$$

We will need the following lemma.

Lemma 4.2 *Let $\mathfrak{a}$ be a proper fractional $\mathcal{O}_f$-ideal. Then the map*

$$\phi : (\mathfrak{a}^{-1} \setminus \{0\})/\mathcal{O}_f^\times \longrightarrow \left\{I \in [\mathfrak{a}] : I \subset \mathcal{O}_f\right\}$$

defined by $\phi([\alpha]) = \alpha\mathfrak{a}$ is a bijection.

Proof We first prove that the map ϕ is well-defined. Observe that if $\alpha \in \mathfrak{a}^{-1}$, then $\alpha\mathfrak{a} \subseteq \mathcal{O}_f$ since $\mathfrak{a}^{-1}\mathfrak{a} = \mathcal{O}_f$. Next, observe that if $[\alpha] = [\beta]$, then $\alpha = \beta u$ for some unit $u \in \mathcal{O}_f^\times$. It follows that $\alpha\mathcal{O}_f = \beta u\mathcal{O}_f = \beta\mathcal{O}_f$, and hence $\alpha\mathfrak{a} = \beta\mathfrak{a}$.

To prove that ϕ is injective, suppose that $\alpha\mathfrak{a} = \beta\mathfrak{a}$. Then $\alpha\mathfrak{a}\mathfrak{a}^{-1} = \beta\mathfrak{a}\mathfrak{a}^{-1}$, which implies that $\alpha\mathcal{O}_f = \beta\mathcal{O}_f$, or equivalently, that $[\alpha] = [\beta]$.

To prove that ϕ is surjective, suppose that $I \in [\mathfrak{a}]$ with $I \subset \mathcal{O}_f$. Then $I = \alpha\mathfrak{a}$ for some $\alpha \in K^\times$, or equivalently, $I\mathfrak{a}^{-1} = \alpha\mathcal{O}_f$. Since I is integral, we have $I\mathfrak{a}^{-1} \subset \mathfrak{a}^{-1}$, so that $\alpha \in \mathfrak{a}^{-1}$. Then $[\alpha] \in (\mathfrak{a}^{-1} \setminus \{0\})/\mathcal{O}_f^\times$ with $\phi([\alpha]) = \alpha\mathfrak{a} = I$. □

We now prove Proposition 4.1.

Proof of Proposition 4.1 Using Lemmas (4.2) and (4.1), we get

$$\begin{aligned}
\zeta_{[\mathfrak{a}]}(s) = \sum_{\substack{I\in[\mathfrak{a}]\\ I\subset\mathcal{O}_f}} \frac{1}{N(I)^s} &= \sum_{0\neq\alpha\in\mathfrak{a}^{-1}/\mathcal{O}_f^\times} \frac{1}{N(\alpha\mathfrak{a})^s} \\
&= \frac{1}{N(\mathfrak{a})^s} \sum_{0\neq\alpha\in\mathfrak{a}^{-1}/\mathcal{O}_f^\times} \frac{1}{N(\alpha)^s} \\
&= \frac{1}{a^s} \sum_{0\neq\alpha\in\mathfrak{a}^{-1}/\mathcal{O}_f^\times} \frac{1}{|\alpha|^{2s}} \\
&= \frac{1}{w_f a^s} \sum_{(0,0)\neq(m,n)\in\mathbb{Z}^2} \frac{1}{|m + nz_{\mathfrak{a}^{-1}}|^{2s}}
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{w_f}\left(\frac{\sqrt{|\Delta_f|}}{2}\right)^{-s}\left(\frac{\sqrt{|\Delta_f|}}{2a}\right)^{s}\sum_{(0,0)\neq(m,n)\in\mathbb{Z}^2}\frac{1}{|m+nz_{\mathfrak{a}^{-1}}|^{2s}}\\
&= \frac{1}{w_f}\left(\frac{\sqrt{|\Delta_f|}}{2}\right)^{-s}\sum_{(0,0)\neq(m,n)\in\mathbb{Z}^2}\frac{\mathrm{Im}(z_{\mathfrak{a}^{-1}})^s}{|m+nz_{\mathfrak{a}^{-1}}|^{2s}}.\\
&= \frac{2}{w_f}\left(\frac{\sqrt{|\Delta_f|}}{2}\right)^{-s}\zeta(2s)E(z_{\mathfrak{a}^{-1}},s),
\end{aligned}
$$

where for the last equality we used the following well-known identity (see e.g. [4, Proposition 2.7.6 (a), p. 55])

$$
\zeta(2s)E(z,s) = \frac{1}{2}\sum_{(0,0)\neq(m,n)\in\mathbb{Z}^2}\frac{\mathrm{Im}(z)^s}{|m+nz|^{2s}}.
$$

□

5 A Chowla–Selberg formula for imaginary quadratic orders

In this section, we will prove the following result.

Theorem 5.1 *With notation as in Sect. 4, we have*

$$
\prod_{[\mathfrak{a}]\in\mathrm{Cl}(\mathcal{O}_f)} F(z_{\mathfrak{a}^{-1}}) = \left(\frac{1}{4\pi\sqrt{|\Delta_f|}}\right)^{h(\mathcal{O}_f)/2}\prod_{k=1}^{|D|}\Gamma\left(\frac{k}{|D|}\right)^{\chi_D(k)w_D h(\mathcal{O}_f)/4h(D)} \times \prod_{p|f} p^{-e(p)h(\mathcal{O}_f)/2},
$$

where $F(z)$ is defined by (3.2), $z_{\mathfrak{a}^{-1}}$ is a CM point as in (4.1), and

$$
e(p) := \frac{(1-p^{\mathrm{ord}_p(f)})(1-\chi_D(p))}{p^{\mathrm{ord}_p(f)-1}(1-p)(\chi_D(p)-p)}. \tag{5.1}
$$

Before proving Theorem 5.1, we illustrate how it can be used to evaluate the Dedekind eta function $\eta(z)$ at CM points.

Example 5.2 Let $K = \mathbb{Q}(i)$, and consider the order of conductor $f = 2$ in K given by

$$
\mathcal{O}_2 = \mathbb{Z} + 2\mathbb{Z}[i] = \mathbb{Z} + \mathbb{Z}2i.
$$

Since the discriminant of K is $D = -4$, the discriminant of $\mathcal{O}_2$ is $\Delta_2 = 2^2(-4) = -16$. Also, $h(-4) = 1$ and $w_{-4} = 4$. We have $h(\mathcal{O}_2) = 1$, so that $\mathrm{Cl}(\mathcal{O}_2) = \{[\mathcal{O}_2]\}$. Then since $\mathcal{O}_2^{-1} = \mathcal{O}_2 = \mathbb{Z} + \mathbb{Z}2i$, from (4.1) we can take $z_{\mathcal{O}_2^{-1}} = 2i$ for the CM point. It follows that

$$
\prod_{[\mathfrak{a}]\in\mathrm{Cl}(\mathcal{O}_2)} F(z_{\mathfrak{a}^{-1}}) = F(z_{\mathcal{O}_2^{-1}}) = F(2i) = \sqrt{\mathrm{Im}(2i)}|\eta(2i)|^2 = \sqrt{2}|\eta(2i)|^2.
$$

On the other hand, we have

$$\left(\frac{1}{4\pi\sqrt{|\Delta_2|}}\right)^{h(\mathcal{O}_2)/2}\prod_{k=1}^{4}\Gamma\left(\frac{k}{4}\right)^{\chi_{-4}(k)w_{-4}h(\mathcal{O}_2)/4h(-4)}\prod_{p|2}p^{-e(p)h(\mathcal{O}_2)/2}$$
$$=\frac{1}{4\sqrt{\pi}}\prod_{k=1}^{4}\Gamma\left(\frac{k}{4}\right)^{\chi_{-4}(k)}2^{-e(2)/2}.$$

Therefore, by Theorem 5.1 we get

$$\sqrt{2}|\eta(2i)|^2=\frac{1}{4\sqrt{\pi}}\prod_{k=1}^{4}\Gamma\left(\frac{k}{4}\right)^{\chi_{-4}(k)}2^{-e(2)/2}. \tag{5.2}$$

Now, the values of the Kronecker symbol are $\chi_{-4}(1)=1$, $\chi_{-4}(2)=0$, $\chi_{-4}(3)=-1$, and $\chi_{-4}(4)=0$. The only prime $p|f$ is $p=2$, and we have

$$e(2)=\frac{(1-2)(1-\chi_{-4}(2))}{2^{1-1}(1-2)(\chi_{-4}(2)-2)}=-\frac{1}{2}.$$

Then after expanding the product in (5.2), we get

$$\sqrt{2}|\eta(2i)|^2=\frac{1}{2^{7/4}\pi^{1/2}}\frac{\Gamma(1/4)}{\Gamma(3/4)}. \tag{5.3}$$

Using the reflection formula

$$\Gamma(z)\Gamma(1-z)=\frac{\pi}{\sin(\pi z)}$$

with $z=1/4$ yields

$$\Gamma(3/4)=\frac{2\pi}{\sqrt{2}}\frac{1}{\Gamma(1/4)}.$$

Then substituting in (5.3) gives

$$|\eta(2i)|^2=\frac{1}{2^{11/4}\pi^{3/2}}\Gamma(1/4)^2.$$

Finally, since $\eta(2i)$ is a positive real number, we get

$$\eta(2i)=\frac{1}{2^{11/8}\pi^{3/4}}\Gamma(1/4). \tag{5.4}$$

We used `SageMath` [14] to compute that both sides of (5.4) are ≈0.592382781332416, which serves as a numerical verification of Theorem 5.1.

Proof of Theorem 5.1 By Proposition 4.1, we have

$$\zeta_{[\mathfrak{a}]}\left((s+1)/2\right)=\frac{2}{w_f}\left(\frac{\sqrt{|\Delta_f|}}{2}\right)^{-(s+1)/2}\zeta(s+1)E\left(z_{\mathfrak{a}^{-1}},(s+1)/2\right).$$

Then summing over all ideal classes in $\mathrm{Cl}(\mathcal{O}_f)$ yields

$$\zeta_{\mathcal{O}_f}\left((s+1)/2\right) = \frac{2}{w_f}\left(\frac{\sqrt{|\Delta_f|}}{2}\right)^{-(s+1)/2} \zeta\left(s+1\right) \sum_{[\mathfrak{a}]\in \mathrm{Cl}(\mathcal{O}_f)} E\left(z_{\mathfrak{a}^{-1}}, (s+1)/2\right). \tag{5.5}$$

For convenience, define the function

$$g_{\mathcal{O}_f}(s) := \frac{w_f}{2}\left(\frac{\sqrt{|\Delta_f|}}{2}\right)^{(s+1)/2} \frac{\zeta_{\mathcal{O}_f}\left((s+1)/2\right)}{\zeta\left(s+1\right)}.$$

Then (5.5) can be written as

$$g_{\mathcal{O}_f}(s) = \sum_{[\mathfrak{a}]\in \mathrm{Cl}(\mathcal{O}_f)} E\left(z_{\mathfrak{a}^{-1}}, (s+1)/2\right). \tag{5.6}$$

Now, using the Kronecker limit formula (3.1), we compare Taylor expansions at $s=-1$ on both sides of (5.6) to get

$$g'_{\mathcal{O}_f}(-1) = \sum_{[\mathfrak{a}]\in \mathrm{Cl}(\mathcal{O}_f)} \log(F(z_{\mathfrak{a}^{-1}})),$$

or equivalently,

$$\prod_{[\mathfrak{a}]\in \mathrm{Cl}(\mathcal{O}_f)} F(z_{\mathfrak{a}^{-1}}) = \exp[g'_{\mathcal{O}_f}(-1)]. \tag{5.7}$$

Therefore, we must evaluate $g'_{\mathcal{O}_f}(-1)$.

Our starting point is the factorization (see e.g. [1, Proposition 10.18 (2)])

$$\zeta_{\mathcal{O}_f}(s) = \zeta(s)L_f(s)L(\chi_D, s),$$

where

$$L_f(s) := \prod_{p|f} \frac{(1-p^{-s})(1-\chi_D(p)p^{-s}) - p^{\mathrm{ord}_p(f)(1-2s)-1}(1-p^{1-s})(\chi_D(p)-p^{1-s})}{1-p^{1-2s}}.$$

We use this factorization to write

$$g_{\mathcal{O}_f}(s) = \frac{w_f}{2}\left(\frac{\sqrt{|\Delta_f|}}{2}\right)^{(s+1)/2} \frac{\zeta\left((s+1)/2\right)}{\zeta\left(s+1\right)} L_f\left((s+1)/2\right) L\left(\chi_D, (s+1)/2\right).$$

Now, a calculation with the product rule yields

$$g'_{\mathcal{O}_f}(-1) = \frac{w_f}{4} L_f(0)L(\chi_D, 0)\left(\log\left(\frac{\sqrt{|\Delta_f|}}{2}\right) - \frac{\zeta'(0)}{\zeta(0)} + \frac{L'(\chi_D, 0)}{L(\chi_D, 0)} + \frac{L'_f(0)}{L_f(0)}\right).$$

To further simplify this identity, we note that

$$L_f(0) = f\prod_{p|f}\left(1 - \frac{\chi_D(p)}{p}\right).$$

Then using Dirichlet's class number formula

$$L(\chi_D, 0) = \frac{2h(D)}{w_D}, \tag{5.8}$$

the identity (see e.g. [3, Theorem 7.24])

$$h(\mathcal{O}_f) = \frac{h(D)f}{[\mathcal{O}_K^\times : \mathcal{O}_f^\times]} \prod_{p|f} \left(1 - \frac{\chi_D(p)}{p}\right),$$

and $[\mathcal{O}_K^\times : \mathcal{O}_f^\times] = w_D/w_f$, we get

$$\frac{w_f}{4} L_f(0) L(\chi_D, 0) = \frac{h(\mathcal{O}_f)}{2}.$$

It follows that

$$g'_{\mathcal{O}_f}(-1) = \frac{h(\mathcal{O}_f)}{2} \left(\log\left(\frac{\sqrt{|\Delta_f|}}{2}\right) - \frac{\zeta'(0)}{\zeta(0)} + \frac{L'(\chi_D, 0)}{L(\chi_D, 0)} + \frac{L'_f(0)}{L_f(0)} \right). \tag{5.9}$$

We now evaluate the logarithmic derivatives of $\zeta(s)$, $L(\chi_D, s)$, and $L_f(s)$ at $s = 0$. Using the special values $\zeta(0) = -1/2$ and $\zeta'(0) = -\log(2\pi)/2$, we get

$$\frac{\zeta'(0)}{\zeta(0)} = \log(2\pi). \tag{5.10}$$

Next, consider the decomposition

$$L(\chi_D, s) = |D|^{-s} \sum_{k=1}^{|D|} \chi_D(k) \zeta(s, k/|D|), \tag{5.11}$$

where

$$\zeta(s, x) := \sum_{n=0}^{\infty} \frac{1}{(n + x)^s}, \quad x > 0, \quad \mathrm{Re}(s) > 1$$

is the Hurwitz zeta function. Lerch [11] proved that

$$\zeta(s, x) = \frac{1}{2} - x + \log\left(\frac{\Gamma(x)}{\sqrt{2\pi}}\right) s + O(s^2). \tag{5.12}$$

We then substitute (5.12) into (5.11), differentiate, and use (5.8) to get

$$\frac{L'(\chi_D, 0)}{L(\chi_D, 0)} = -\log(|D|) + \frac{w_D}{2h(D)} \sum_{k=1}^{|D|} \chi_D(k) \log\Gamma\left(\frac{k}{|D|}\right). \tag{5.13}$$

Finally, we evaluate the logarithmic derivative of $L_f(s)$ at $s = 0$. For convenience, write

$$L_f(s) = \prod_{p|f} \frac{G_p(s)}{H_p(s)},$$

where

$$G_p(s) := (1-p^{-s})(1-\chi_D(p)p^{-s}) - p^{\mathrm{ord}_p(f)(1-2s)-1}(1-p^{1-s})(\chi_D(p)-p^{1-s})$$

and $H_p(s) := 1 - p^{1-2s}$. Then

$$\frac{L_f'(s)}{L_f(s)} = \frac{d}{ds}\log(L_f(s)) = \sum_{p|f}\left(\frac{G_p'(s)}{G_p(s)} - \frac{H_p'(s)}{H_p(s)}\right). \tag{5.14}$$

Now, we have

$$\begin{aligned} G_p'(s) = &(1-p^{-s})\log(p)\chi_D(p)p^{-s} + (1-\chi_D(p)p^{-s})\log(p)p^{-s} \\ &+ 2\,\mathrm{ord}_p(f)\log(p)p^{\mathrm{ord}_p(f)(1-2s)-1}(1-p^{1-s})(\chi_D(p)-p^{1-s}) \\ &- p^{\mathrm{ord}_p(f)(1-2s)-1}(\chi_D(p)-p^{1-s})\log(p)p^{1-s} \\ &- p^{\mathrm{ord}_p(f)(1-2s)-1}(1-p^{1-s})\log(p)p^{1-s}. \end{aligned}$$

Hence

$$\frac{G_p'(0)}{G_p(0)} = \log(p)\frac{1-\chi_D(p)+2\,\mathrm{ord}_p(f)p^{\mathrm{ord}_p(f)-1}(1-p)(\chi_D(p)-p) - p^{\mathrm{ord}_p(f)}(\chi_D(p)-p) - p^{\mathrm{ord}_p(f)}(1-p)}{-p^{\mathrm{ord}_p(f)-1}(1-p)(\chi_D(p)-p)}.$$

Also, $H_p'(s) = 2\log(p)p^{1-2s}$ so that

$$\frac{H_p'(0)}{H_p(0)} = \log(p)\frac{2p}{1-p}.$$

From these calculations, we get

$$\begin{aligned} &\frac{1}{\log(p)}\left(\frac{G_p'(0)}{G_p(0)} - \frac{H_p'(0)}{H_p(0)}\right) \\ &= \frac{1-\chi_D(p)+2\,\mathrm{ord}_p(f)p^{\mathrm{ord}_p(f)-1}(1-p)(\chi_D(p)-p) - p^{\mathrm{ord}_p(f)}(\chi_D(p)-p) - p^{\mathrm{ord}_p(f)}(1-p)}{-p^{\mathrm{ord}_p(f)-1}(1-p)(\chi_D(p)-p)} \\ &\quad - \frac{2p}{1-p} \\ &= -2\,\mathrm{ord}_p(f) + \frac{1-\chi_D(p) - p^{\mathrm{ord}_p(f)}(\chi_D(p)-p) - p^{\mathrm{ord}_p(f)}(1-p) + 2p^{\mathrm{ord}_p(f)}(\chi_D(p)-p)}{-p^{\mathrm{ord}_p(f)-1}(1-p)(\chi_D(p)-p)} \\ &= -2\,\mathrm{ord}_p(f) - \frac{(1-p^{\mathrm{ord}_p(f)})(1-\chi_D(p))}{p^{\mathrm{ord}_p(f)-1}(1-p)(\chi_D(p)-p)}. \end{aligned} \tag{5.15}$$

Then substituting (5.15) into (5.14) yields

$$\begin{aligned}\frac{L_f'(0)}{L_f(0)} &= \sum_{p|f} -\log(p)\left(2\operatorname{ord}_p(f) + \frac{(1-p^{\operatorname{ord}_p(f)})(1-\chi_D(p))}{p^{\operatorname{ord}_p(f)-1}(1-p)(\chi_D(p)-p)}\right)\\ &= \log\left(\prod_{p|f} p^{-\left(2\operatorname{ord}_p(f)+\frac{\left(1-p^{\operatorname{ord}_p(f)}\right)(1-\chi_D(p))}{p^{\operatorname{ord}_p(f)-1}(1-p)(\chi_D(p)-p)}\right)}\right)\\ &= \log\left(\left(\prod_{p|f} p^{\operatorname{ord}_p(f)}\right)^{-2}\prod_{p|f} p^{-e(p)}\right)\\ &= \log\left(f^{-2}\prod_{p|f} p^{-e(p)}\right), \end{aligned} \tag{5.16}$$

where $e(p)$ is defined by (5.1).

To complete the evaluation of $g'_{\mathcal{O}_f}(-1)$, we substitute (5.10), (5.13), and (5.16) into (5.9) and use $|\Delta_f| = f^2|D|$ to get

$$\begin{aligned} g'_{\mathcal{O}_f}(-1) &= \frac{h(\mathcal{O}_f)}{2}\log\left(\left(\frac{\sqrt{|\Delta_f|}}{4\pi f^2|D|}\right)\prod_{k=1}^{|D|}\Gamma\left(\frac{k}{|D|}\right)^{\chi_D(k)w_D/2h(D)}\prod_{p|f} p^{-e(p)}\right)\\ &= \log\left(\left(\frac{1}{4\pi\sqrt{|\Delta_f|}}\right)^{h(\mathcal{O}_f)/2}\prod_{k=1}^{|D|}\Gamma\left(\frac{k}{|D|}\right)^{\chi_D(k)w_D h(\mathcal{O}_f)/4h(D)}\prod_{p|f} p^{-e(p)h(\mathcal{O}_f)/2}\right), \end{aligned}$$

or equivalently,

$$\exp[g'_{\mathcal{O}_f}(-1)] = \left(\frac{1}{4\pi\sqrt{|\Delta_f|}}\right)^{h(\mathcal{O}_f)/2}\prod_{k=1}^{|D|}\Gamma\left(\frac{k}{|D|}\right)^{\chi_D(k)w_D h(\mathcal{O}_f)/4h(D)}\prod_{p|f} p^{-e(p)h(\mathcal{O}_f)/2}.$$

By (5.7), this completes the proof. □

6 Faltings heights of CM elliptic curves

In this section, we will prove the following result which is based on Silverman [16, Proposition 1.1].

Proposition 6.1 *Let E/L be an elliptic curve with complex multiplication by an order $\mathcal{O}_f$ in an imaginary quadratic field K. Assume that the coefficients of the Weierstrass equation for E/L are contained in $\mathbb{Q}(j(E))$. Then*

$$h_{\mathrm{Fal}}(E/L) = \frac{\log(N_{L/\mathbb{Q}}(\Delta_{E/L}))}{12[L:\mathbb{Q}]} - \log(2\pi) - \frac{1}{h(\mathcal{O}_f)}\sum_{[\mathfrak{a}]\in \mathrm{Cl}(\mathcal{O}_f)}\log(F(z_{\mathfrak{a}^{-1}})),$$

where $F(z)$ is defined by (3.2) and $z_{\mathfrak{a}^{-1}}$ is a CM point as in (4.1).

Proof Given $\sigma \in \mathrm{Hom}(L,\mathbb{C})$, let $z_\sigma \in \mathbb{H}$ be a complex number such that

$$E^\sigma(\mathbb{C}) \cong \mathbb{C}/(\mathbb{Z}+\mathbb{Z}z_\sigma). \tag{6.1}$$

Moreover, let

$$\Delta(z) := (2\pi)^{12}\eta(z)^{24}$$

be the discriminant function. Then Silverman [16, Proposition 1.1] proved that the Faltings height of E/L is given by

$$h_{\mathrm{Fal}}(E/L) = \frac{\log(N_{L/\mathbb{Q}}(\Delta_{E/L}))}{12[L:\mathbb{Q}]} - \frac{1}{12[L:\mathbb{Q}]}\sum_{\sigma\in\mathrm{Hom}(L,\mathbb{C})}\log(\mathrm{Im}(z_\sigma)^6|\Delta(z_\sigma)|). \tag{6.2}$$

Note that

$$\frac{1}{12}\log(\mathrm{Im}(z_\sigma)^6|\Delta(z_\sigma)|) = \log(2\pi) + \log(F(z_\sigma)),$$

hence (6.2) can be written as

$$h_{\mathrm{Fal}}(E/L) = \frac{\log(N_{L/\mathbb{Q}}(\Delta_{E/L}))}{12[L:\mathbb{Q}]} - \log(2\pi) - \frac{1}{[L:\mathbb{Q}]}\sum_{\sigma\in\mathrm{Hom}(L,\mathbb{C})}\log(F(z_\sigma)). \tag{6.3}$$

Now, write

$$\sum_{\sigma\in\mathrm{Hom}(L,\mathbb{C})}\log(F(z_\sigma)) = \sum_{\tau\in\mathrm{Hom}(\mathbb{Q}(j(E)),\mathbb{C})}\ \sum_{\substack{\sigma\in\mathrm{Hom}(L,\mathbb{C})\\ \sigma|_{\mathbb{Q}(j(E))}=\tau}}\log(F(z_\sigma)).$$

Since E/L has coefficients in $\mathbb{Q}(j(E))$, then for each fixed $\tau\in\mathrm{Hom}(\mathbb{Q}(j(E)),\mathbb{C})$ we can take the same point $z_\sigma\in\mathbb{H}$ in the isomorphism (6.1) for all $\sigma\in\mathrm{Hom}(L,\mathbb{C})$ such that $\sigma|_{\mathbb{Q}(j(E))}=\tau$. Therefore, if we let $\sigma_\tau\in\mathrm{Hom}(L,\mathbb{C})$ denote any of the $[L:\mathbb{Q}(j(E))]$ embeddings which extend $\tau\in\mathrm{Hom}(\mathbb{Q}(j(E)),\mathbb{C})$, then we have

$$\sum_{\tau\in\mathrm{Hom}(\mathbb{Q}(j(E)),\mathbb{C})}\ \sum_{\substack{\sigma\in\mathrm{Hom}(L,\mathbb{C})\\ \sigma|_{\mathbb{Q}(j(E))}=\tau}}\log(F(z_\sigma)) = \sum_{\tau\in\mathrm{Hom}(\mathbb{Q}(j(E)),\mathbb{C})}[L:\mathbb{Q}(j(E))]\log(F(z_{\sigma_\tau})).$$

By Shimura [15, Theorem 7.6], we have $[\mathbb{Q}(j(E)):\mathbb{Q}] = h(\mathcal{O}_f)$ and

$$\{j(E)^\tau:\ \tau\in\mathrm{Hom}(\mathbb{Q}(j(E)),\mathbb{C})\} = \{j(\mathfrak{a}^{-1}):\ [\mathfrak{a}]\in\mathrm{Cl}(\mathcal{O}_f)\}.$$

Then for each $\tau\in\mathrm{Hom}(\mathbb{Q}(j(E)),\mathbb{C})$, there is a unique $[\mathfrak{a}]\in\mathrm{Cl}(\mathcal{O}_f)$ such that $E^{\sigma_\tau}(\mathbb{C})\cong\mathbb{C}/\mathfrak{a}^{-1}$. Recalling that $\mathfrak{a}^{-1}=\mathbb{Z}+\mathbb{Z}z_{\mathfrak{a}^{-1}}$ (see (4.1)), we get

$$\mathbb{C}/(\mathbb{Z}+\mathbb{Z}z_{\sigma_\tau})\cong\mathbb{C}/(\mathbb{Z}+\mathbb{Z}z_{\mathfrak{a}^{-1}}),$$

hence the points z_{σ_τ} and $z_{\mathfrak{a}^{-1}}$ are $SL_2(\mathbb{Z})$-equivalent (see e.g. [17, Proposition I.4.4]). Since $F(z)$ is $SL_2(\mathbb{Z})$-invariant, it follows that

$$\sum_{\tau\in\mathrm{Hom}(\mathbb{Q}(j(E)),\mathbb{C})}[L:\mathbb{Q}(j(E))]\log(F(z_{\sigma_\tau})) = \frac{[L:\mathbb{Q}]}{h(\mathcal{O}_f)}\sum_{[\mathfrak{a}]\in\mathrm{Cl}(\mathcal{O}_f)}\log(F(z_{\mathfrak{a}^{-1}})).$$

Finally, the preceding calculations yield

$$\frac{1}{[L:\mathbb{Q}]}\sum_{\sigma\in\mathrm{Hom}(L,\mathbb{C})}\log(F(z_\sigma)) = \frac{1}{h(\mathcal{O}_f)}\sum_{[\mathfrak{a}]\in\mathrm{Cl}(\mathcal{O}_f)}\log(F(z_{\mathfrak{a}^{-1}})),$$

which by (6.3) completes the proof. □

7 Proofs of Theorem 1.1 and Theorem 1.3

In this section, we prove Theorems 1.1 and 1.3.

Proof of Theorem 1.1 By Proposition 6.1, we have

$$h_{\mathrm{Fal}}(E/L) = \log\left(N_{L/\mathbb{Q}}(\Delta_{E/L})^{1/12[L:\mathbb{Q}]}(2\pi)^{-1}\prod_{[\mathfrak{a}]\in \mathrm{Cl}(\mathcal{O}_f)} F(z_{\mathfrak{a}^{-1}})^{-1/h(\mathcal{O}_f)}\right). \tag{7.1}$$

Moreover, by Theorem 5.1 we have

$$\prod_{[\mathfrak{a}]\in \mathrm{Cl}(\mathcal{O}_f)} F(z_{\mathfrak{a}^{-1}})^{-1/h(\mathcal{O}_f)} = \left(\frac{1}{4\pi\sqrt{|\Delta_f|}}\right)^{-1/2}\prod_{k=1}^{|D|}\Gamma\left(\frac{k}{|D|}\right)^{-\chi_D(k)w_D/4h(D)}\prod_{p|f} p^{e(p)/2}. \tag{7.2}$$

Then by substituting (7.2) into (7.1) and simplifying, we obtain Theorem 1.1. □

Proof of Theorem 1.3 Since E/L has complex multiplication, the j-invariant $j(E)$ is an algebraic integer. Hence by [18, Proposition VII.5.5], E/L has potential good reduction. Accordingly, let L'/L be a finite extension such that E/L' has everywhere good reduction. Now, by [18, Proposition III.1.4], there is a finite extension L''/L' with $\mathbb{Q}(j(E)) \subset L''$ and an elliptic curve E/L'' such that E/L'' is given by a Weierstrass equation with coefficients in $\mathbb{Q}(j(E))$ and such that E/L' is isomorphic to E/L''. By the semistable reduction theorem [[18], Proposition VII.5.4 (b)], the curve E/L'' also has everywhere good reduction. Therefore we have $\Delta_{E/L''} = \mathcal{O}_{L''}$. Finally, since $N_{L''/\mathbb{Q}}(\Delta_{E/L''}) = 1$, then Theorem 1.3 follows by applying Theorem 1.1 to E/L'' and observing that

$$h_{\mathrm{Fal}}^{\mathrm{stab}}(E/L) = h_{\mathrm{Fal}}(E/L') = h_{\mathrm{Fal}}(E/L'').$$

□

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